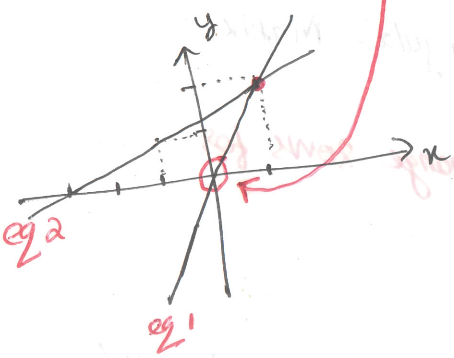
Linear Algebra

Geometry of Linear Equations

- Row picture → One equation at a time
- Column picture → linear combination of column vectors
- Matrix form

eg. $2x - y = 0$
$-x + 2y = 3$

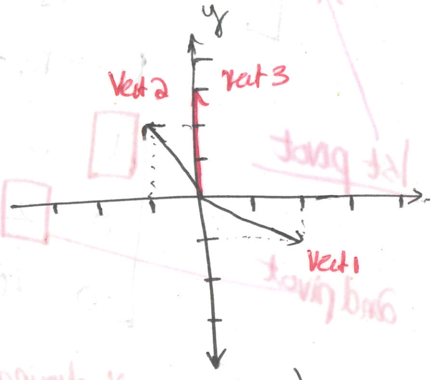
Row picture



Solution $x=1, y=2$

Column picture

$$x\begin{array}{c}\text{Vect 1}\\ \begin{bmatrix}2\\-1\end{bmatrix}\end{array} + y\begin{array}{c}\text{Vect 2}\\ \begin{bmatrix}-1\\2\end{bmatrix}\end{array} = \begin{array}{c}\text{Vect 3}\\ \begin{bmatrix}0\\3\end{bmatrix}\end{array}$$



Combine the vectors in the right amount (Linear Combination) of column vectors.

$1(\text{Vect } 1) + 2(\text{Vect } 2) = \text{Vect } 3$

$x=1$ $y=2$

Solution.

- Any vector can be formed by the linear combination of vectors 1 & 2.
- The linear combination of vectors 1 & 2 spans the entire x,y plane.

Matrix form $Ax=b$

$$\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

Conclusions

- Can I solve $Ax=b$ for every b?
- When I multiply a matrix by a vector, I get a linear combination of column
- If ~~two~~ all columns are linearly independent it can have any b for which a linear combination exists.
- If any column ~~thes~~ ~~in~~ is a linear combination of other columns, then this column vector would line in same plane as other vectors.
  └ (plane in 3 dimension
  n-1 surface in n^{th} dimension)
- Eg, in a 9 dimension vector space, if the colums are not independent then it cannot be solved for all b.
- if only say m are linearly independent & other 9-m are Linear Comb of these m vectors. then solution only for b which exist in m dimension plane

Elimination of Variables with Matrices

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \qquad \begin{aligned} x+2y+z &= 2 \\ 3x+8y+z &= 12 \\ 4y+z &= 2 \end{aligned} \qquad (AX=b)$$

1st pivot

Multiplier

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

2nd pivot

3rd pivot.

x eliminated from remaining equation

Upper triangular Matrix

→ Pivot's can't be zero — If pivot becomes zero exchange rows for non zero pivot.

→ Determinant = Product of Pivot

Case of Failure

→ If any of the pivot is zero, the matrix equation (After elimination) is not solvable ∀ all b.

→ The determinant is zero

Solution

- $x+2y+z=2$
  $2y-2z=6$
  $5z=-10$

  equivalent system of equations after transforming to upper triangle.

- Matrix times column vector = linear combination of Matrix column vector
- Row vector times Matrix = linear combination of Matrix Rows

eg:-

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \times \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}}_{(E_{21}A)}$$

Multiplier

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}}_{E_{32}}\underbrace{\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}}_{(E_{21}A)} = \underbrace{\begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}}_{U}$$

↳ elimentary matrices

E_{32} $(E_{21}A)$ = U

Permutation Matrix (Exchange rows)

eg. exchange Rows 1 & 2 $\longrightarrow \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

- To find a matrix that if multiplied with a given matrix say A would be equalent to performing an elementary row operation on A, all you have to do is to perform the same elementary operation of the identify matrix.
- Similarly for columns....

Multiplying by putting elementary matrix on left → Row operation

Multiplying by putting elementary matrix on right → Column operation.

E_{32} | E_{31} | E_{21} | A → Upper triangular matrix

Permutation matrix if needed | No permutation needed. | Permutation matrix if needed

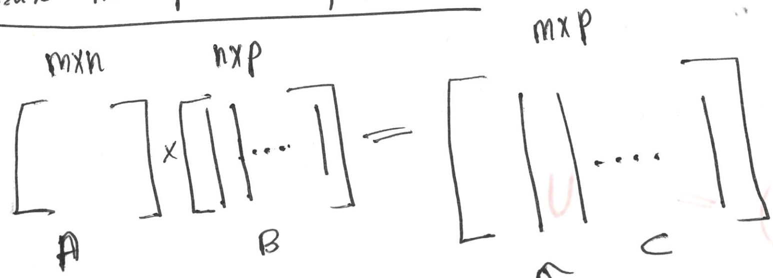
Inverses

A^{-1} → Add three time row 1 to row 2

A → Sub three times row 1 from row 2

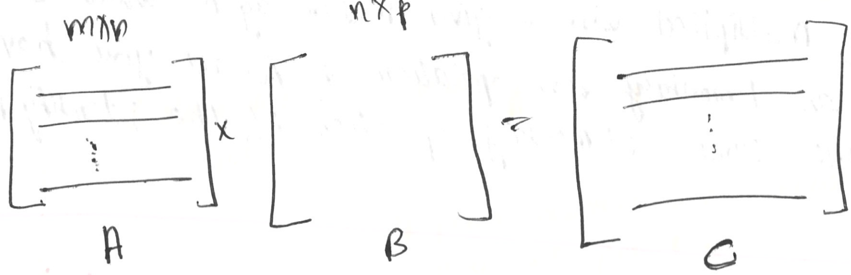
$$\underbrace{\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I}$$

Matrix multiplication & Inverses



Columns of C are linear combination of column of A

A single column of C is the matrix A multiplied by a corresponding column in B



A single row of C is the corresponding row of A multiplied by B.

Similarly rows of C are combination of rows of B

- $AB = \text{Sum of } (i^{th} \text{ column of } A \times i^{th} \text{ row of } B)$

eg:- $\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$

Row space :- ~~All the~~ Vector space spaned by the Linear combination of rows of a matrix.

Column space: Vector space spaned by the linear combination of rows of a matrix.

Block multiplication

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]}_{B} = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$$

! valid

Inverses [square matrices]

$A^{-1}A = I = AA^{-1}$ (If A^{-1} exist)

invertible, non singular

Q) When is a matrix not invertible.

eg:- $\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}}$

Approach 1

From knowledge of determinant, we can see that the determinant is zero & hence non invertible

Approach 2

Columns of matrix A are multiples

the ~~vectors~~ column vectors line in a straight line and the column space is only this straight line and the column combination of column vectors of A ~~can~~ and hence no combination of column vectors of A can generate $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to create a identity matrix.

Approach 3

$Ax = 0$ can be solved for $X \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ There is a non zero vector X such that $AX = 0$

here $x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

From definition of linear independence $a_1 c_1 + a_2 c_2 = 0$ only for $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

if $Ax = 0$ & A^{-1} exist

$A^{-1}AX = A^{-1}0$

$IX = 0 \Rightarrow X = 0$

if $X \neq 0$ assumption wrong

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \quad A^{-1} \quad = I$$

$$A \begin{bmatrix} \text{Column } J \text{ of} \\ A^{-1} \end{bmatrix} = \begin{bmatrix} \text{Column } J \text{ of} \\ I \end{bmatrix}$$ — n set of linear system of equations.

Gauss Jordan (solve 2 equations at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Augmenting 2 constant matrices

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{A \quad\quad I} \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{I \quad\quad A^{-1}}$$

proof

$$E_{eff} [A \mid I] = [I \mid ?]$$

$$[E_{eff} A \mid E_{eff} I] = [I \mid ?]$$

$\Rightarrow E_{eff} A = I$ | $\Rightarrow E_{eff} I = ?$

$E_{eff} = A^{-1}$ | $\Rightarrow ? = A^{-1}$

Inverse of AB

$$AA^{-1} = I = A^{-1}A$$

$$(AB)(AB)^{-1} = A(BB^{-1})A^{-1} = I$$

Inverse of A^T

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

$$AA^{-1} = I$$

$$(AA^{-1})^T = I = (A^{-1})^T A^T$$

inverse of A^T

★ ⟹ Order of Transpose & Inverse are interchangable.

A = LU (No row Exchanges)

• L - lower triangular

• U - upper ~~diagonal~~ triangular

$$\begin{array}{cc} E_{21} & A \\ \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \end{array} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

Pivots.

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

(elementary) elimination matrices can be inverted by negating the off diagonal elements if there are no row exchanges.

3x3 (Assuming no row exchanges)

$$EA = U$$

$$E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U$$

$$\cancel{A = E_{32}^{-1} E_{31}^{-1} E_{21}^{-1}}$$

$$A = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1} \cdot U$$

$$A = LU$$

• The multipliers go directly into L.

eg:- E_{21}

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{31} = I$

E_{32}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$E_{32} \cdot E_{31} \cdot E_{21} = E$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{31}^{-1} = I$

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

A can be expressed as a product of lower & upper triangular matrices if there are no row exchanges.

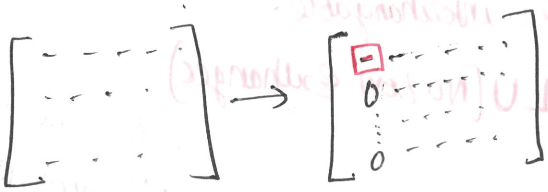
$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Cost for Elimination

How many operations on a $n \times n$ matrix A?

let $n = 100$

$\times$ Counting Row transformation is counted as 1 operation.



$\checkmark$ each time a element is updated is taken as 1 operation

1 column cleared $\rightarrow 99 \times 100 \sim 100^2$

2 column cleared $\rightarrow 99 \times 98 \sim 99^2$

$\vdots$

$n^2 + (n-1)^2 + (n-2)^2 \sim n^3$

$\therefore$ $\boxed{\begin{array}{l} \text{No. of operation Big O } n^3 \\ \text{No. of operations } \approx \frac{1}{3}n^3 \end{array}}$

Cost of B $\sim n^2$

Permutations 3x3

(Matrix that exchanges rows)

eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q) How many 3x3 permutation matrices?

Ans — 6.

No exchanges — 1 only identity.

Two row exchanges — 3

All rows changed — 2

$\left.\begin{array}{l} 1 \\ 3 \\ 2 \end{array}\right\}$ Total 6 P's.

Set of all permutation matrices — P

$\boxed{P_i \cdot P_j = P_k}$

$\boxed{P_n^{-1} = P_m^{-1}}$

$\boxed{P^{-1} = P^T}$

$\star$ The transposes, inverses and products of permulation matrices are still in the group P.

Q) How many 4x4 permutation matrices 24.

$\boxed{\text{General } n!}$

Permutation P : Execute Row exchanges

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cdot & 1 & 0 & 0 \\ \cdot & \cdot & 1 & \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ & 1 & \cdot & \cdot \\ & 0 & 1 & \cdot \\ & & & 1 \end{bmatrix}$$ becomes $PA = LU$ — Any invertible matrix A

Assuming no row exchange
assuming $P = I$

Permutations P is the identity matrix with reordered rows

$P^{-1} = P^T$

$P^T P = I$

$n!$ permutation matrices are possible for n^{th} order matrix

eg: $n = 3 \Rightarrow 6$
$n = 4 \Rightarrow 24$
$n = 5 \Rightarrow 120$

Transposes

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = R$$

General formula

$$\boxed{(A^T)_{ij} = A_{ji}}$$

Symmetric Matrices

$A^T = A$

$R^T R$ = Symmetric ~~$R R^T$~~

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & \ldots \\ 7 & 11 & \ldots \end{bmatrix}$$

$(R^T R)^T = \cancel{R^T R}$

$R^T \cdot R^{TT} = R^T \cdot R$

Vector Spaces

Requirements: if v, w are any two vectors in a vector space, the the linear combination $c_1v + c_2w$ must also lie in the vector space.

Examples : R^2 = all 2-dimensional real vectors like $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \dots$

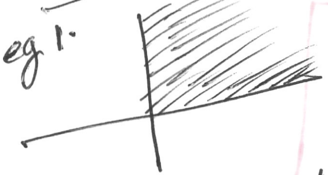
= xy plane

R^3 = all 3-dimensional real vectors

R^n = all vectors with n components which are real

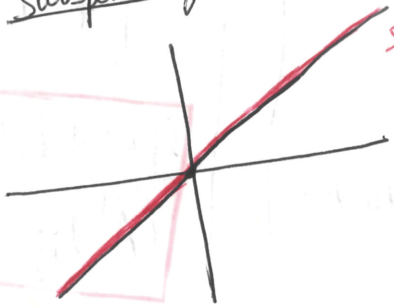
Not a vector space

eg 1.

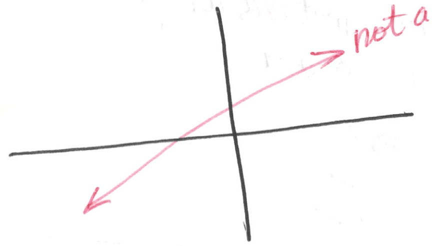


multiplication is not closed

Subspace eg (A vector space inside R_2)



subspace is a line $\boxed{\text{through origin}}$



not a subspace of R^2

Subspaces of R^2

1. All of R_2
2. Any line through origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. zero vector only

Subspaces of R^3

1. All of R_3
2. Any line through origin
3. Plane through origin
4. zero vector only

Subspace from matrices

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

Columns are in R^3 : Column Space $C(A)$

all their linear combination form a subspace

→ In this case Plane through origin which has the vectors column 1 vector & column 2 vector.

Subspaces S_1 and S_2

$\boxed{\star\star\star\ S_1 \cap S_2 \text{ is also a subspace}}$

$\boxed{\text{Column space of A}}$: Subspace of R^4 : All linear combinations of columns

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{4\times 3}$$

Q Does $Ax=b$ always have a solution for every b?

Ans No.

Q) which b's allow the system to be solved.

eg $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

I can solve $Ax=b$ exactly when b is in the column space.

$\boxed{\text{Null Space of a vector A}}$: All solutions of vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of $Ax = 0$

eg: $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ here null space is a subspace of R^3

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- one solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ vector independent of A.
- another solution is $x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} k \\ k \\ -k \end{bmatrix}$

$$Ax = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

~~another solution~~

$x = k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

subspace of R^3
Null space of A is a line through origin.

Check that Solutions to $Ax=0$ always give a subspace.

Requirement: If x^1 is a solution & x^2 is a solution then $c_1x^1+c_2x^2$ is also a solution.

$$\left.\begin{aligned} Ax^1&=0 \\ Ax^2&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} A(x^1+x^2)&=0 \\ Ax^1+Ax^2&=0 \\ 0+0&=0 \end{aligned}$$ proof!

Q) $Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ Do all the solution $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ form a vector space.

because $X=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not a solution & hence the solution cannot be a vector space.

$X=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution, $X=\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is a solution

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Finding Solution of $AX=0$

$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$

Row transformations
- Null space not changed
- Column space changed.

$A' = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ 2-pivots

$U = A'' = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ echelon form! Rank=2

Rank = No. of pivots
= No. of non zero rows

2 pivot columns

Other columns → free column

Therefore

$x = \begin{bmatrix} 2k_1-2k_2 \\ k_2 \\ -2k_1 \\ k_1 \end{bmatrix}$

free column

$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

pivot column

We would have to assume $(n-r)$ variables when n = no. of unknowns

$$x_1+2x_2+2x_3+2x_4=0$$
$$2x_3+4x_4=0$$

let $x_4=k_1$
$\Rightarrow x_3=-2k_1$

let $x_2=k_2$

$$x_1+2x_2+-4k_1+2k_1=0$$
$$x_1+2x_2-2k_1=0 \Rightarrow x_1=2k_1-2k_2$$

$$x = k_1 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \text{defines the null space}$$

Rank defines the no. of independent rows or columns whichever is lowest

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n = no. of unknowns only.

$n-r$ defines the max no. of independent vectors that can be acheived in the null space where n = no. of unknowns.

Reduced Row echelon form (zeros above & below pivots, pivots are ones.)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{2}} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice I matrix in pivot rows & colums

$RX = 0$

$$\begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$x_1 \quad k_2 \quad x_3 \quad k_1$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 0 & 2 & -2 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$x_1 = -2k_2 + 2k_1$

$x_2 = k_2$

$x_3 = -2k_1$

$x_4 = k_1$

r = rank

$$R = \left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} r \text{ pivot rows} \\ \\ \end{array}$$

r pivot colums $\quad$ $(n-r$ free colums$)$

null space matrix $\rightarrow$ columns are special solution $\therefore [X] \neq 0$

where $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

eg 2

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[U]$

$r = 2$ ~~rank~~ $n - r = 1$ (1 independent vector in null space)

free column

null space changes for transpose

$$\boxed{\text{Rank of } A = \text{Rank of } A^T \rightarrow \text{same}}$$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 2x_3 = 0$

set $x_3 = k$

$x_2 = -k$

$x_1 + 2(-k) + 3(k) = 0$

$x_1 + k = 0$

$x_1 = -k$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [X] = k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{array}{c} I \\ \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \equiv \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

null space matrix $\underline{\underline{\quad}}$ solution

$$\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solving $Ax=b$

$$A= \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Augmented Matrix $= [A\, b]$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

Solution exists for $\forall\ b_3 = b_1 + b_2$

Solvability: Condition on b

- $Ax=b$ is solvable if b is in the column space of matrix A.
- If the combination of rows of A gives the zero row, then the same combination of the ~~tombu~~ components/entries of $[b]$ must given zero.
- To find the complete solution to $Ax=b$

let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ $\therefore$ Solution exists.

① $X_{particular}$: Set the free variables to zero
: Solve $Ax=b$ for pivot variables

$$\left.\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_3 &= 3/2 \\ x_1 &= -2 \end{aligned}\right\} \quad X_{particular} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

② X_{null_space} (X_n)

$$\boxed{\text{Complete solution} = X_{particular} + X_{null\text{-}space}}$$

$$A X_p = b$$
$$A X_n = 0$$
$$\overline{A(X_p + X_n) = b}$$

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Plot all solutions X in $\mathbb{R}^4$

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \underbrace{k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{2 dimensional subspace}}$$

doesn't contain zero

overall not a subspace but a shifted plane of a subspace.

---

m by n matrix of rank r $(r \leq m)$; $(r \leq n)$

m – no. of equations
n – no of unknowns.

I Case Full Column Rank

$r = n \Rightarrow$ No free variables

Nullspace of A $N(A) = \{\text{zero vector}\}$

$\therefore$ Solution to $Ax = b$ $X_{complete} = X_{particular}$

$\therefore$ System has unique solution if it exists.

eg: $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$

Reduced row echelon form

$A' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$Ax = 0$ for $x = 0$ only.

II Full Row Rank

$(r = m) \Rightarrow m$ pivots.

Can solve $Ax = b$ which RHS.

for every b because no zero rows

$n - m$ $(n - r)$ free variables

eg:- $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$

Row reduced echelon form $R = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & F & \end{bmatrix}$

III $r = m = n$ square matrix $\Rightarrow$ invertible.

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R → Row reduced echelon form

☆ $N(A) = 0$

☆ b_1, b_2 can be anything

Summary

$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$
$R = I$	$R = [\, I \;\; F \,]$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution for $\forall b$	infinitely many solution $\forall b$	0 or 1 solution $\forall b$	(0 or ∞ solutions $\forall b$)

Suppose A is m by n matrix $m < n$

Then there are non zero solutions X to the eq. $AX = 0$

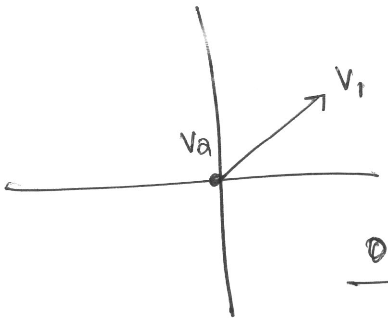
→ More unknowns than equations.

Independance

Vectors $x_1, x_2, \ldots, x_n$ are independent if no combination gives the zero vectors except the zero combination.

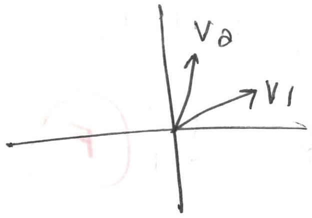
$c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = 0$ only for $(c_1, c_2, c_n = 0)$

$V_2 =$ zero vector



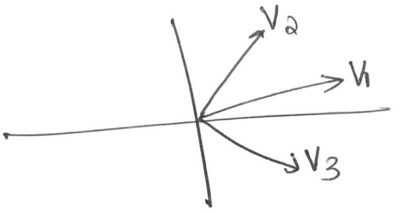
$\underline{0}\, V_1 + \underline{K}\, V_2 = 0$

$\therefore$ dependent



$\underline{\quad} V_1 + \underline{\quad} V_2 = 0$

independent.



dependent

$$\begin{array}{c} V_1 \quad V_2 \quad V_3 \\ \left[\begin{array}{ccc} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{array} \right] \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 V_1 + x_2 V_2 + x_3 V_3 = 0$

solution exists

because $R = \begin{bmatrix} I & F \end{bmatrix}$

Vectors $V_1, V_2, V_3 \ldots V_n$ are independent if the nullspace of A is only the zero vector. Where matrix A is formed by taking $V_1, V_2, V_3 \ldots V_n$ ~~are~~ as columns of A. They are dependent if $N(A)$ is a non zero subspace which means non zero solution exist for $Ax = 0$ $\rightarrow$ Non zero linear combination of $V_1 \ldots V_n$ is zero

independent rank $(A) = n$ } n is no of variables
dependent rank $(A) < n$ } n is no. of columns.

* span = Create a vector space by taking linear combination of certain vectors.

Basis for a vector space S is a set of vectors $V_1, V_2 \ldots V_n$ with 2 properties

1) They are independent
2) They span the vector space S

Example:

Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

Another basis is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \cancel{\begin{bmatrix}3\\3\\8\end{bmatrix}}$

n vectors $\{v_1, v_2, \dots v_n\}$ for a basis of $\mathbb{R}^n$ if the matrix $A_{n\times n}$ formed by vectors $v_1, v_2 \dots v_n$ as columns is invertible $\Rightarrow rank(A) = n$
$N(A) = [0]_{n\times 1}$

* Every basis for a vector space S has a the same no. of vectors

The no. of vectors in the basis of a vector space S is called the dimension of vector space S.

eg: Column space (A)

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 1 & 1 & 2 & 1\\ 1 & 2 & 3 & 1\end{bmatrix}$$

Do the column vectors span the column space _ Yes.
Are they a basis for the column space _ No.

$Ax = 0$ $x = \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$

Q. Find the basis of the column space
column vectors 1 2 2 $\begin{bmatrix}1 & 2\\ 1 & 1\\ 1 & 2\end{bmatrix}$,

Another basis $\begin{bmatrix}1 & 3\\ 1 & 2\\ 1 & 3\end{bmatrix}, \begin{bmatrix}2 & 3\\ 1 & 2\\ 2 & 3\end{bmatrix}, \begin{bmatrix}5 & 6\\ 3 & 4\\ 5 & 6\end{bmatrix}$

rank (A) = No. of pivot columns = Dimension of the column space

Dimension of Nullspace: $Dim(N(A)) = n - r$

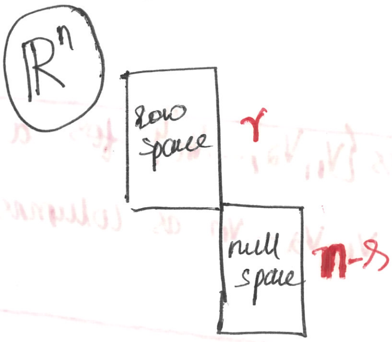
n = no. of variables

4 Fundamental Subspaces

1) Column space $C(A)$
2) Null space $N(A)$
3) Row space $R(A)$ = column space of (A^T)

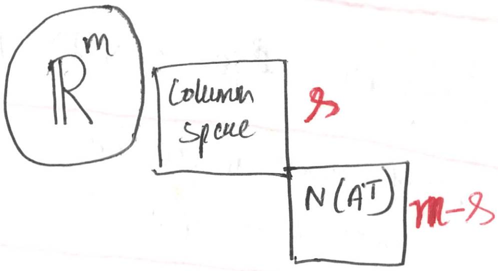
$\boxed{R(A) = C(A^T)}$

4) Null space of A^T $N(A^T)$ (left null space of A)



A is m×n

$C(A)$ is in $\mathbb{R}^m$
$N(A)$ is in $\mathbb{R}^n$
$C(A^T)$ is in $\mathbb{R}^n$
$N(A^T)$ is in $\mathbb{R}^m$



basis? Dimension?

Dimension of column space $C(A) = r$

	$C(A)$	$R(A) = C(A^T)$	$N(A)$	$N(A^T)$
Basis	Pivot column	~~[illegible]~~ Pivot rows	Trivial solution	
Dimension	rank(A)	rank (A)	n − rank(A)	m − rank(A)

*A is m×n
**rank(A) = r

eg: $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

R - Reduced row echelon form

elimentary row operation

⇒ Different column space
But same row space

$N(A^T)$

$A^T y = 0$

$(A^T y)^T = 0$

$y^T A = 0$

$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rref}\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m}\end{bmatrix}$ $\Big\}$ $EA = R$, $EI = E$ ★★★

$\Rightarrow E_{m\times m}\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} \longrightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m}\end{bmatrix}$

$\boxed{\text{if } A \text{ is invertible } R = I \text{ then } E \text{ is } A^{-1}}$

---

eg: $\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} = \begin{bmatrix} 1&2&3&1&1&0&0\\ 1&1&2&1&0&1&0\\ 1&2&3&1&0&0&1 \end{bmatrix} \rightarrow \begin{bmatrix} 1&2&3&1&1&0&0\\ 0&-1&-1&0&-1&1&0\\ 0&0&0&0&-1&0&1 \end{bmatrix} \rightarrow \begin{bmatrix} 1&2&3&1&1&0&0\\ 0&1&1&0&1&-1&0\\ 0&0&0&0&-1&0&1 \end{bmatrix}$

$\downarrow$

$\left[\begin{array}{cccc|ccc} 1&0&1&1&-1&2&0\\ 0&1&1&0&1&-1&0\\ 0&0&0&0&-1&0&1 \end{array}\right]$

$[R \mid E]$ left null space $N(A^T)$

Very important technique.

check.

$E = \begin{bmatrix} -1&2&0\\ 1&-1&0\\ -1&0&1 \end{bmatrix}$

$EA = \begin{bmatrix} -1&2&0\\ 1&-1&0\\ -1&0&1 \end{bmatrix} \times \begin{bmatrix} 1&2&3&1\\ 1&1&2&1\\ 1&2&3&1 \end{bmatrix} = \begin{bmatrix} 1&0&1&1\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix}$

left null space $N(A^T)$

New Vector space (M)

all 3x3 matrix

$\Big\}$ A matrix qualifies as a Vector because

- addition of matrix is possible
- scallable by scalar
- linear combination of matrix
- zero matrix exist
- Additive & multiplicative identity exist.

A+B & cA is valid
not intersted in AB for now

Subspaces of M

→ All upper triangular
→ All symmetric matrices
→ diagonal matrices — dimension is = 3

Basis of M = all 3x3's

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\text{Dimension} = 9.}$$

1) Subspace of M ① Symmetric 3x3 = S
dimension of S = 6

2) Subspace of M ② Upper triangular 3x3 = U
dimension of S = 6

$S \cap U$ = Set of all 3x3 diagonal matrices

dimension of $S \cap U = 3$

$S + U$ = any element of S + any element in U = all 3x3 = M

$S + U \neq S \cup U$

$S + U \to$ span

dimension of $S + U = 9$

dim $S = 6$

dim $U = 6$

dim $S \cap U = 3$

General formula

* $\boxed{\dim A + \dim B = \dim A \cap B + \dim A + B}$

Q) $\frac{d^2y}{dx^2} + y = 0$ solution to ODE $y = \cos x$, $y = \sin x$, $y = e^{ix}$

Complete solution $y = c_1 \sin x + c_2 \cos x$

Basis = $\sin x, \cos x$

RANK one matrices

dim $C(A)$ = rank = dim $C(A^T) = 1$

eg. $A_{2\times3} = \begin{bmatrix}1&4&5\\2&8&10\end{bmatrix} = \underset{2\times1}{\begin{bmatrix}1\\2\end{bmatrix}} \underset{1\times3}{\begin{bmatrix}1&4&5\end{bmatrix}}$

$\boxed{\text{Rank1 matrix has the form } UV^T}$

Q) $M =$ all 5×17 matrices

Subset of rank 4 matrices or less

will this form a vector space

No because when we add two rank 4 matrices, we may get a rank 5 matrice.

Q) In R^4 $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ let $S =$ all vectors $V \in R^4$ with $v_1+v_2+v_3+v_4=0$ is S a subspace? If so dimension & basis.

Solution

Yes it is a subspace. Dimension $\Rightarrow$

$$v_1+v_2+v_3+v_4=0$$
$$\Rightarrow AV=0$$
$$A=[1\ 1\ 1\ 1]$$
$$V=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$[1\ 1\ 1\ 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}=0$$

$S =$ nullspace of A.

rank of $A = 1$

$n - r = 3$

$\therefore$ Dimension $= 3$

Basis $= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

row space of $A =$ span $[1\ 1\ 1\ 1] \in R^4$
1 dim

column space of $A = [1] \in R^1$

$C(A) = R^1$

$N(A^T) = \{0\}$

$\dim C(A) = 1 \quad \in R^1$

$\dim N(A) = 3 \quad \in R^4$

$\dim C(A^T) = 1 \quad \in R^4$

$\dim N(A^T) = 0 \quad \in R^1$

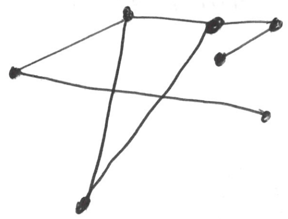
$n=4$

$m=1$

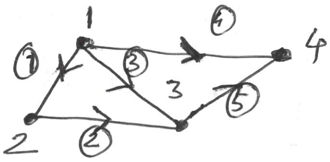
Small world graphs

$$\text{Graph} = \{\text{nodes}, \text{edges}\}$$

eg.



Graphs & Networks



$n = 4$ nodes
$m = 5$ edges

Incidence matrix $A =$

$$\begin{array}{c|c}
 & \overbrace{\begin{matrix}1 & 2 & 3 & 4\end{matrix}}^{\text{nodes}} \\
\end{array}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \left.\begin{matrix}1\\2\\3\\4\\5\end{matrix}\right\} \text{edges}$$

Finding null space

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \text{potential at nodes}$$

Ax = potential differences.

$$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\dim N(A) = 1$

Basis $= c\,[1\ 1\ 1\ 1]^T$

rank $(A) = 3$

$A^T y = 0$ (Kirchoff's current Law)

$\dim N(A^T) = m - r$
$= 5 - 3 = 2$

$y^T = [y_1\ y_2\ y_3\ y_4\ y_5]$

$$A^T y = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Row1 $\times y \Rightarrow -y_1 - y_3 - y_4 = 0$ (KCL @ node 1)

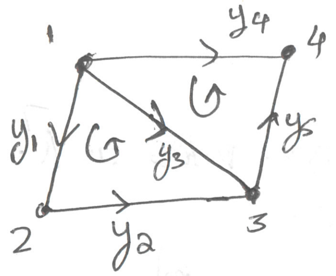
Row2 $\times y$ = KCL @ node2

Row $n \times y$ = KCL @ node n

Basis of $N(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

KCL satisfying loop



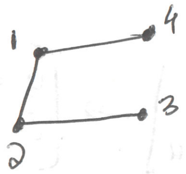
outer loop

$$y = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

[not independent on basis]

$C(A^T)$ - Row space

Column 1,2,3 are not independent in A^T as they form a loop.



Independent columns don't follow loop

dependent columns form loop.

graph with no loops but all nodes — Tree

dim $N(A^T) = m - r$ = # no. of independent loops

r = rank (A)

rank $= n - 1$

rank = # nodes - 1

loops = # edges -

loops independent = # edges - (# nodes - 1)

(# nodes) - (# edges) + (# loops) = 1

Euler's formula

y = branch current matrix

x = potential ~~difference~~ matrix @ node

$A^T y = 0 \rightarrow$ KCL with no external current sources

$A^T y = f \rightarrow f$ represent current sources

$Ax = e \rightarrow$ potential difference matrix 'e', Ax

$y = Ce$ (ohm's law)

$C \rightarrow$ Admittance

Summary

$$\left.\begin{aligned} Ax &= e \\ y &= Ce \\ A^T y &= f \end{aligned}\right\} \Rightarrow \quad y = CAx$$

$$\boxed{A^T C A x = f}$$

Review

Q) Suppose u, v, w are non zero vectors in $\mathbb{R}^7$. What are the possible dimensions if they span a vector space.

Solution) 1, 2, 3 (not 0 as non zero vectors) (not >3 as only 3 vectors)

Q) 5x3 Matrix - U in R echelon form with 3 pivots
$r = 3$
a) What is the null space

Solution
Nullspace $= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

if $B = \begin{bmatrix} U \\ 2U \end{bmatrix}$ what is the Reduced echelon form?

~~$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$~~

Reduction

$\begin{bmatrix} U \\ 0 \end{bmatrix}$

Rank = 3

if $C = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} U & U \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$

Rank = 6

$\dim N(C^T)$ Solution $C = 10 \times 6$
$m = 10$
$\dim = 10 - 6 = \underline{\underline{4}}$

$Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ $\quad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

a) $\dim C(A^T)$?

we know $\dim N(A) = 2$ from the general solution.
$= n - r = 2$ but $n = 3$ because $N(A) \in \mathbb{R}^n$
$\Rightarrow r = 1$

$\dim C(A^T) = m - r$ but $m = 3$ from the b $\quad A_{m\times n} X_{n\times 1} = b_{m\times 1}$

$\Rightarrow \dim C(A^T) = 3 - 1 = \underline{\underline{2}}$

b) Find A.

$\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$\Rightarrow c_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & c_2 & c_3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow c_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & \\ 2 & -2 & c_3 \\ 1 & -1 & \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

c) what vectors b can $Ax=b$ be solved? / for any b in $C(A)$

$$b = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

---

a) $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4} = \begin{bmatrix} & & \\ & 3\times 4 & \\ & & \end{bmatrix}$

c = invertible

basis for null space

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$B = CDx = 0$

$\Rightarrow C^{-1}CDx = C^{-1}0 = 0$

$= Dx = 0$

D is in Row reduced form

$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $N(D) = \begin{bmatrix} -F \\ 6 \end{bmatrix}$

---

b) Solve $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in prev question.

because 1'st column of B is the ~~particu~~ b.

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Q) If A, B has the same four subspaces then $A = cB$ True or False

Solution A & B are two different invertible matrix eg 6×6 then

$C(A) = R^6$ They have same rank

$C(A^T) = R^6$

$N(A) = \{[0]\}$

$N(A) = \{[0]\}$

Q) If two rows of a matrix are interchanged, which spaces stay same

1) Row space.

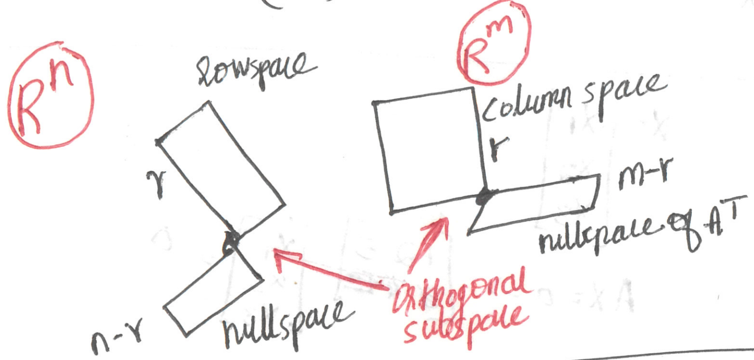
2) Null space.

Q) Why can't the vector $[1, 2, 3]$ be in the row & also be ~~a~~ in null space.

Orthogonal Vectors & Subspaces

nullspace $\perp$ to row space

$$N(A^TA) = N(A)$$



orthogonal $\equiv$ perpendicular

1) $X^TY = 0$ test for orthogonals.

2) $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

is there a connection?

---

length of a vector $= \sqrt{X^TX}$

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $X+Y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|x\|^2 = 14$ $\|y\|^2 = 5$ $\|x+y\|^2 = 19$

$\downarrow$ $\downarrow$ $\downarrow$

X^TX Y^TY $(x+y)^T(x+y)$

$$\cancel{X^TX} + \cancel{Y^TY} \overset{?}{=} \cancel{X^TX} + X^TY + Y^TX + \cancel{Y^TY}$$

$$X^TY + Y^TX \overset{?}{=} 0$$

$$= 2X^TY = 0$$

$$\Rightarrow \boxed{X^TY = 0}$$

* zero vector is orthogonal to everybody

---

* Subspace S is orthogonal to subspace T means: Every vector in S is orthogonal to every vector in T

Q) Row space is $\perp$ to null space, why?

X such that: $AX = 0$ (null space)

space spanned by rows (row space)

$$\begin{bmatrix} \text{row 1 of } A \\ \text{row 2 of } A \\ \dots \text{row n of } A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

ROW1 . X = 0
ROW2 . X = 0
⋮
ROW N . X = 0

$\{ROW1^T, ROW2^T, ROW N^T\}$
row space $\perp$ to X.

Row space $(R) = a_1(\text{ROW}_1)^T + a_2(\text{ROW}_2)^T + a_3(\text{ROW}_3)^T + \dots a_n(\text{ROW}_n)^T$

$R^T X = a_1(\text{ROW}_1)X + a_2\text{ROW}_2 X + \dots a_n \text{ROW}_n X = 0$

Hence Rowspace(R) ⊥ to nullspace X

$A = \begin{bmatrix} 1 & 2 & 5 \\ \cancel{2} & \cancel{4} & \cancel{10} \end{bmatrix}$ $n=3$, $r=1$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

dim Rspace = 1 (line)

dim Null space = 2 (plane)

$AX = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ \cancel{2} & \cancel{4} & \cancel{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

* nullspace & rowspace are orthogonal
* Their dimensions add up to whole dimension
* Orthogonal complements in R^n
* orthogonal complement to a space contains all vectors ⊥ to it.
* Nullspace contains ~~of~~ all vectors ⊥ to rowspace

★★★

Comming up: Solve $Ax = b$ if there are no solutions ie there is noise in b.

— done all the time

eg Scenario #1 • Measurement of position of satlite m no # of times

• using m measurements to find out n parameters.

• m equations
n parameters

Scenario #2 • Measurement of pulse rate m times, only 1 unknown. ie the pulse rate

A — $m \times n$ $(m>0)$
m # equations
n # unknowns

Measurements have noise in RHS

Best solution
• Eliminate noise from actual information
• Looking for best fit

1) • Throw away equation until you get invertible system
• Not best solution as it rejects data.

Matrix A^TA

$(A^TA)^T = A^T(A^T)^T = A^TA$

- $(n\times m)\times(m\times n) = n\times n \rightarrow$ square
- Symmetric

Multiplying $Ax=b$ by A^T on both sides

$\Rightarrow A^TA\hat{x} = A^Tb$

eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$

$m>n$ | (r) Rank $=2$
$3>2$

$Ax=b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

* (A^TA may not be always invertible)

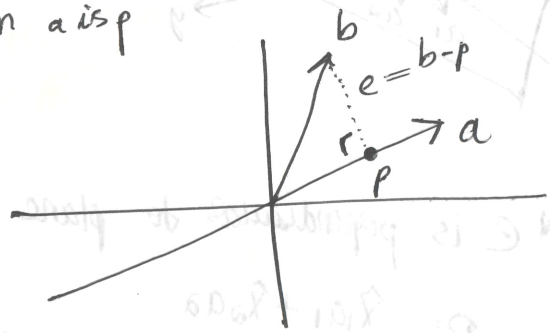
~~Null space of A^TA~~

eg. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$ not invertible

Null space of (A^TA) = Nullspace of A
Rank of (A^TA) = Rank A
A^TA is invertible exactly when null space of A is 0 vectors
ie A has independent columns.

Projections

projection of b on a is p



~~1) On line~~

$p = xa$ (some multiple of a)

x – scalar

$a \perp$ to e

$e = (b - xa)$

$\Rightarrow a^T(b - xa) = 0$ (since perpendicular)

$x\, a^Ta = a^Tb$

$$x = \frac{a^Tb}{a^Ta}$$

$$X = \frac{a \cdot b}{||a||^2} = \frac{a^T b}{a^T a} \qquad p = x a$$

$$p = \boxed{a \cdot \left|\frac{a^T b}{a^T a}\right|}$$

projection $p = \mathbb{P} b$ → projection matrix.

a, b – vectors

$$\mathbb{P} = \frac{a a^T}{a^T a}$$

$a^T a$ — (length of $a)^2 = ||a||^2$

Properties of P

* rank of $\mathbb{P} = 1$
* column space of $\mathbb{P}$ = subspace created by line a
* $P^T = P$ they are symmetric.
* $P \cdot (Pb) = Pb$

$\Rightarrow P^2 = P$

$\Rightarrow P^n = P$

why project?

Because $Ax=b$ may have no solution

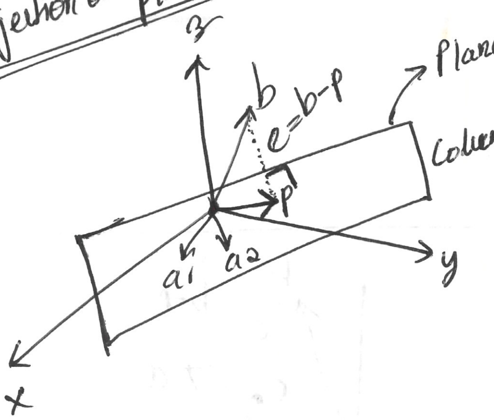
b maynot be in the column space of A (hence no solution)

Solve $A\hat{x} = p$ instead

where $p = Pb$

↳ Best fit that can be solved.

2) Projection on plane



$p = Pb$

$(a_1, a_2) \in$ subspace of plane 1

$(a_1, a_2) \in$ column space

column space $\left[A = [a_1 \; a_2] \right]$

* e is perpendicular to plane

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$$\boxed{p = A\hat{x}} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ x_2 \end{bmatrix} = \hat{x}_1 \begin{vmatrix} \\ a_1 \\ \\ \end{vmatrix} + \hat{x}_2 \begin{vmatrix} \\ a_2 \\ \\ \end{vmatrix}$$

solving for $A\hat{x} = p$ instead for $Ax = b$

$p = A\hat{x}$: Find $\hat{x}$?

key: $p \perp e$

$\Rightarrow a_1 \perp (b - A\hat{x})$

$a_2 \perp (b - A\hat{x})$

$\Rightarrow a_1^T(b - A\hat{x}) = 0$

$a_2^T(b - A\hat{x}) = 0$

$$\begin{bmatrix} - a_1^T - \\ - a_2^T - \end{bmatrix} \begin{bmatrix} | \\ b - A\hat{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A^T

$\Rightarrow A^T(b - A\hat{x}) = 0$

e: $\Rightarrow A^Te = 0$

e is in $N(A^T)$

e is the null space of A^T

e is all orthogonal vector to column space of A

$\Rightarrow A^Tb - A^TA\hat{x} = 0$

$\boxed{A^TA\hat{x} = A^Tb}$ $\rightarrow$ Find $\hat{x}$ then ($A\hat{x} = p \rightarrow$ best fit for Nonsolvable system $Ax = b$)

Projection Matrix

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb$

$\frac{aa^T}{a^Ta}$ — Used to be a scalar

$p = \mathbb{P}b$

$(A^TA)^{-1} \approx \frac{1}{a^Ta}$

$\mathbb{P} = A(A^TA)^{-1}A^T$

(exists only if nullspace of A is zero vector ~~column space~~ Columns of A are independent)

$\mathbb{P} = AA^{-1}(A^T)^{-1}A^T$ — if only ~~★★★★~~

! A is square & invertible

But if A is square & invertible then b is already in column space $\mathbb{R}^n$

$\therefore$ Projection matrix $= I$

Properties of $\mathbb{P}$

1) $\mathbb{P}^T = \mathbb{P}$

2) $\mathbb{P}^2 = \mathbb{P}$

Proof

$\mathbb{P}\cdot\mathbb{P}$

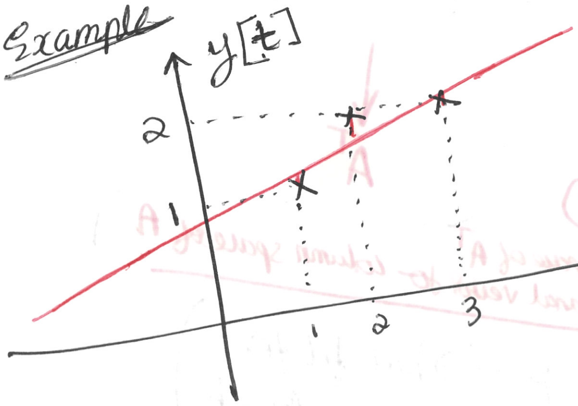
$((A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A)) = (A(A^TA)^{-1}\cancel{(A^TA)}(A^TA)^{-1}A)$

$= A(A^TA)^{-1}A = \mathbb{P}$

Applications

- Least square
- Fitting by Line

Example



Points (1,1), (2,2), (3,2)

assume a best fit line b

$b = C + Dt$

System of equation that cannot be solved

$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$Ax = b$

Instead solving

$A\hat{x} = p$

$A^T A \hat{x} = A^T b$

If b is in column space, $Pb = b$

If b is $\perp$ to column space $Pb = 0$

If b is in null space of (A^T) $Pb = 0$

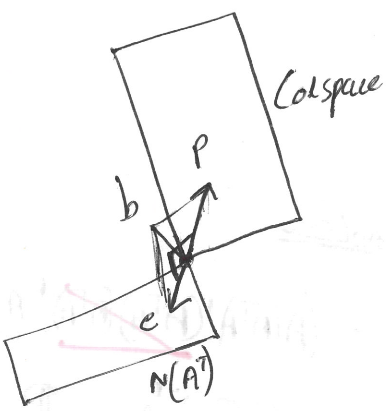
Proof 1) If b is in null space of (A^T) $Pb = 0$

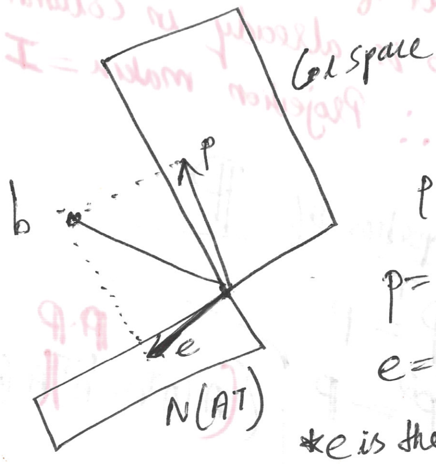
$A(A^TA)^{-1} A^T b$ — if b in $N(A^T)$ then $= 0$.

Proof 2) If b is in column space then $b = Ax$

$A(A^TA)^{-1} A^T [Ax]$

$\Rightarrow A \underline{(A^TA)^{-1}} [A^TA] x = Ax = \underline{\underline{b}}$





$p + e = b$

$p = Pb$

$e = (I - P) b$

*e is the projection on to $\perp$ matrix

Proof

$b = p + e$

$b = \mathbb{P}b + e$

$e = b - \mathbb{P}b$

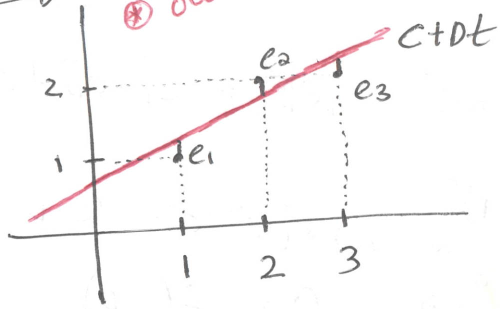
$e = (\underline{I} - \mathbb{P})b$

$e = \mathbb{E}b$

where $\boxed{\mathbb{E} = \mathbb{I} - \mathbb{P}}$

Example again

⊛ outlier



$b_1 = 1$
$b_2 = b_3 = 2$

$e_1 = |p_1 - b_1|$

$e_2 = |p_2 - b_2|$

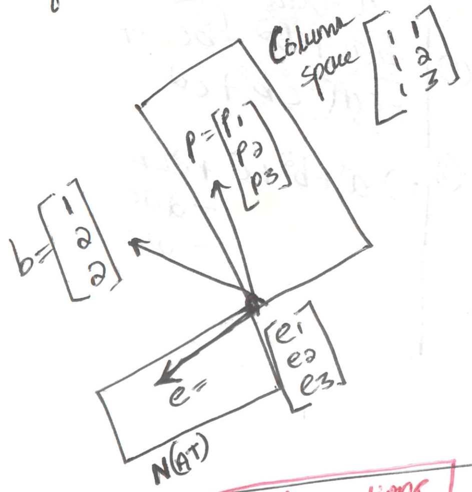
$e_3 = |p_3 - b_2|$

system

$$\left.\begin{matrix} C+D=1 \\ C+2D=2 \\ C+3D=2 \end{matrix}\right\} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Big\} \; Ax = b$$

No solution!

- Little over compensation for outliers



* A has independent columns
* Best possible fit
* $C + D = 1 \pm \text{error}_1$
  $C + 2D = 2 \pm \text{error}_2$
  $C + 3D = 3 \pm \text{error}_3$ $\Rightarrow A\hat{x} = b + e$
* Minimize $\|A\hat{x} - b\|^2 = \|e\|^2 = e_1^2 + e_2^2 + e_3^2$
* Minimise the least square error
* Linear Regression

Q) Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p

1) $\boxed{\text{Normal equations} \quad A^T A \hat{x} = A^T b}$ — most important equation in estimation

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

symmetric & invertible

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{matrix} 5 \\ 11 \end{matrix}$$

$$A^T[A|b] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$$[A^T A \mid A^T b] =$$

$3\hat{C} + 6\hat{D} = 5$
$6\hat{C} + 14\hat{D} = 11$

$R_2 - 2R_1$

$\hat{D} + 2\hat{D} = 1$

$\boxed{\Rightarrow \hat{D} = 1/2}$

$\hat{C} = \frac{5-3}{3} = 2/3$

$$\Rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \hat{x} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$$

Using Calculus to acheive the same result

Minimise $\|A\hat{x}-b\|^2 = \|e\|^2 = e_1^2+e_2^2+e_3^2$

$e_1^2 = (C+D-1)^2$

$e_2^2 = (C+2D-2)^2$

$e_3^2 = (C+3D-2)^2$

$f(e) = e_1^2+e_2^2+e_3^2 = (C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2$

$f(e) = f(C,D)$

→ Take partial derivates of C & = 0

→ Take partial derivates of D & = 0

eg. $f(e) = C^2+D^2+2CD-2C-2D+1\ +$

$C^2+4D^2+4CD-4C-8D+4\ +$

$C^2+9D^2+6CD-4C-12D+4$

$\overline{3C^2+14D^2+12CD-10C-22D+9}$

$f(e) = 3C^2+14D^2+12CD-10C-22D+9$

$\frac{\partial f(e)}{\partial C} = 6C+12D-10 = 0 \Rightarrow 3C+6D = 5$ — ①

$\frac{\partial f(e)}{\partial D} = 28D+12C-22 = 0 \Rightarrow 6C+14D = 11$ — ②

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$[A]\hat{x} = p$

identity

$(a+b-c)^2$

$=(a+b-c)(a+b-c)$

$= a^2+ab-ac+$

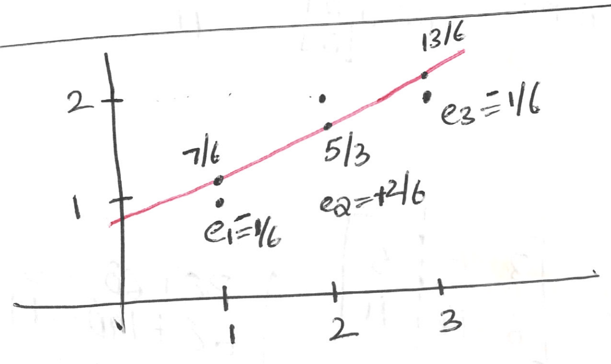
$ba+b^2-bc+$

$-ca-cb+c^2$

$\Rightarrow a^2+b^2+c^2+2ab$

$-2ac$

$-2bc$



$b = p+e$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

$\boxed{p \perp e}$

$p^T e = 0$

$$\begin{bmatrix} 7/6 & 5/3 & 13/6 \end{bmatrix} \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = 0$$

$\|e\|^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{2}{6}\right)^2 + \left(\frac{1}{6}\right)^2$

$\|e\|^2 = \frac{1+4+1}{36} = \frac{6}{36} = 1/6$

Invertible A^TA

* If A has independent columns, then A^TA is invertible.
* If $N(A)$ = {0 vector only}, then columns of A are independent

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow$$

$$c_1x + c_2x_2 = 0$$

Proof

Suppose $A^TAx = 0$

then solution must be $x = 0$ for A^TA to be invertible because $(A^TA)^{-1}A^TAx = (A^TA)^{-1} \cdot 0$

$\Rightarrow I \cdot x = 0$

$\Rightarrow x = 0$

or

$A^TAx = 0$

$x^TA^TAx = 0$

$(Ax)^TAx = 0$

$\Rightarrow \text{length}^2 \; \|Ax\| = 0$

$\Rightarrow Ax = 0 \Rightarrow x = 0$

But $A \neq 0$ & A has independent column vectors

Therefore if we can show that $x=0$ is the only solution, then it is a valid proof.

* Columns are definitely independent if they are perpendicular unit vectors.

orthonormal vectors

normal vectors $\Rightarrow x^Tx = 1$

orthogonal $\Rightarrow x^Ty = y^Tx = 0$

orthonormal ~~matrix~~ vectors x_i only if $x_i^Tx_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Proof

$A^TAx = 0$

$x^TA^TAx = 0$

$(Ax)^TAx = 0$

$\|Ax\|^2 = 0$

$\Rightarrow Ax = 0$

But A has independent column vectors

$\Rightarrow$ Null space = {0 vectors only}

$\Rightarrow x = 0$

Orthogonal Basis & Orthogonal Matrices

orthonormal vectors $(q_i^Tq_j) = \begin{cases} 0 & \text{if } i \neq j \rightarrow \text{ortho} \\ 1 & \text{if } i = j \rightarrow \text{normal} \end{cases}$

$$Q = \begin{bmatrix} \vdots & & \vdots \\ q_1 & \cdots & q_n \\ \vdots & & \vdots \end{bmatrix}$$

ortho normal columns

in matrix form,

$\boxed{Q^TQ = I}$ (orthonormal ~~matrix~~ columns)

Q^TQ = diagonal matrix (orthogonal ~~matrix~~ columns)

If Q is square then $Q^TQ = I$ tells us that $Q^T = Q^{-1}$

* Q is a square matrix with orthonormal columns = orthogonal matrix.

Example:

1)
$$\underset{Q^T}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

2)
$$\underset{Q^T}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \underset{Q}{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

3)
$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4)
$$\underset{Q^T}{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}} \underset{Q}{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5)
$$\underset{3\times 2}{\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}} \cdot \underset{2\times 3}{\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \times$$

$$\downarrow$$

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

$$\Rightarrow \quad Q^TQ = I \not\Rightarrow \boxed{QQ^T = I}$$

If Q square & orthogonal

$Q^T Q = I$

$\Rightarrow$ ~~…~~ orthogonal $\Rightarrow$ columns independent $\Rightarrow$

$\Rightarrow$ rank $= n$

$\Rightarrow$ rows independent

$\Rightarrow$ inverse exist

$\Rightarrow \quad Q^T Q Q^{-1} = Q^{-1}$

$\underline{Q^T = Q^{-1}}$

also

$Q^T Q Q^T = Q^T$

$\underline{(Q^T)^{-1}} Q^T Q Q^T = (Q^T)^{-1} Q^T$

$Q Q^T = I$

$\therefore \boxed{\text{If } Q \text{ is square \& orthogonal then } Q Q^T = Q^T Q = I, \quad Q^T = Q^{-1}}$

* If Q is orthogonal

$\Rightarrow Q^T Q = I \neq Q Q^T$

eg

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \quad \overset{Q^T}{\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{Q}{\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}} \underset{Q^T}{\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Suppose Q has orthonormal columns

Project onto its column space.

$P = Q(Q^T Q)^{-1} Q^T$

$P = Q Q^T$

$Q^T Q = I$

if Q square then $Q Q^T = I$

$\Rightarrow P = I$

Properties

1) Symmetric

2) $(Q Q^T)(Q Q^T) = Q I Q^T = Q Q^T$

$\Rightarrow P P = P$

$P^2 = P$

$P^n = P$

$Q^T Q \hat{x} = Q^T b$

$\hat{x} = Q^T b$

$\boxed{\hat{x}_i = q_i^T b}$

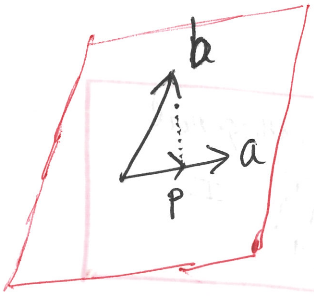
$\longrightarrow$ for orthogonal Q solving for $Q\hat{x} = p$

Gram-Schmidt

Making the colums orthonormal while remaining in column space.

independent vectors $a, b \longrightarrow q_1, q_2$ orthonormal vectors

(orthogonal vectors)



$a, b \rightarrow A, B \longrightarrow q_1, q_2$

(orthonormal vectors)

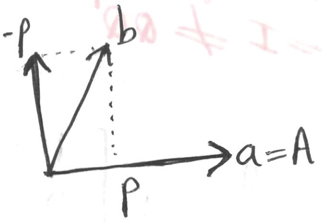
$\boxed{A = a}$

$$\boxed{q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}}$$

$B = b - (b \text{ projection } a)$

$B = b - p = e$

$p = xA$



$$\left.\begin{matrix} p \perp e \\ a \perp e \\ A \perp e \end{matrix}\right\} \Rightarrow A^T e = 0$$

$A^T(b - xA) = 0$

$A^T b - xA = 0$

$A^T b = x A^T A$

$x = \dfrac{A^T b}{(A^T A)}$ since $(A^T A)$ inverse exist since A has independent colunes

$$\Rightarrow \boxed{B = b - \frac{A^T b}{A^T A} A}$$ Gram's formula

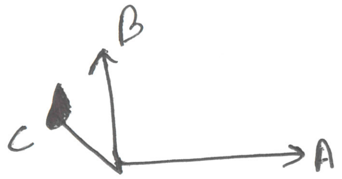
$A \perp B$ proof

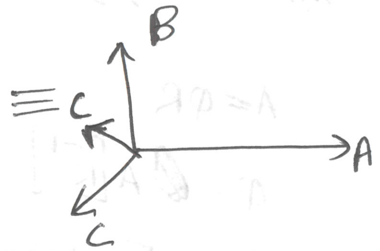
$A^T B$ must be zero

$$A^T B = A^T b - A^T \underbrace{\left(\frac{A^T b}{A^T A}\right)}_{\text{just a number}} A$$

$$= A^T b - \cancel{A^T A} \frac{A^T b}{\cancel{A^T A}} = 0$$

independent $\rightarrow$ abc $\rightarrow$ ABC $\rightarrow$ $q_1 q_2 q_3$





$$C = C - P$$

$$C = C - \frac{A^T C}{A^T A} \cdot A - \frac{B^T C}{B^T B} B$$

[Since A component & B component does not interfere]

$$P = P_A + P_B$$

$$P = \frac{A^T C}{A^T A} \cdot A + \frac{B^T C}{B^T B} \cdot B$$

$C \perp B$

$C \perp A$

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$a = A$

$$B = b - \left(\frac{A^T b}{A^T A}\right) A$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}{A^T A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \left(\frac{1}{3} \cdot (3)\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Ratio

$$B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

If $A \perp B$?

$$A^T \cdot B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

Gram-Schmidt $\rightarrow A \longrightarrow Q$ column echolon transformations without changing column space

$$\begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$A = QR$

$Q = A[R^{-1}]$

$Q = A$[applying column transformations]

$$Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} q_1^T q_1 & q_2^T q_2 \\ q_1^T q_2 & q_2^T q_2 \end{bmatrix}$$

Properties of Determinants

$\det A = |A|$

→ ① $\det A = I$

→ ② Exchange rows → reverse the sign of determinate

These two properties alone gives the determinat of all permulation matrix. $\det P = \pm 1$ $+$even exchange
$-$ odd exchanges

$P \rightarrow$ Permutation matrix

From property ① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

From property ② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

→ (3a) If one of the row is scalled by t then determinant is also scaled by t.

→ (3b) If one of the row can be expressed as sum of 2 numbers for every element then the determinant can be split as shown.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Determinant is linear in rows | Property 3.

4 → If two rows are equal, the determinant = 0

[direct result of property ② because if two row are equal, then exchange these rows to get same matrix so det is same. but rule 2 says sign reverses ⇒ det = 0]

5 → Elementary row transformation will not change the determinant.

result of property ④ & property ③

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} \quad \text{property 3}$$

$R_2 \to R_2 + kR_1$

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k \cdot (0) \quad \text{property 4}$$

$$\Rightarrow \left| R_2 \to R_2 + kR_1 \right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6 → Row of zeroes leads to → $|A| = 0$ result of property 3

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} c-c & d-d \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} + \begin{vmatrix} -c & -d \\ c & d \end{vmatrix} = 0$$

$\boxed{\text{only first 3 rules are defining rules of determinant}}$

7 $$U = \begin{vmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & \cdots & & d_n \end{vmatrix}$$

determinant of upper triangular matrix = product of diagonal elements.

$$= \prod_{j=0}^{n} (d_j) = (d_1 \cdot d_2 \cdot \ldots \cdot d_n)$$

Proof.
① They upper triangular (off diagonal) element doesn't matter because they can made zero. (if $d_1 \ldots d_n \neq 0$) by reduced row echelon form. Assumption

$$\rightarrow \begin{vmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & d_n \end{vmatrix}$$

② property 3
factor out the diagonal element

$$d_1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & & \\ & & \ddots & d_n \end{vmatrix} = d_1 d_2 \ldots d_n \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}$$

③ Property 1 $|I| = 1$

$\Rightarrow |U| = d_1 d_2 d_3 \cdots d_n = \prod_{j=0}^{n} (d_j)$

---

* Case any $d_j = 0$. then elimation results in row of zeroes $= |det| = 0$
So the gap to due assumption is closed.

⑧ → det A = 0 when A is singular

Proof :- If A matrix is singular then after elimination a row of zeroes appears $\Rightarrow |A| = 0$

:- If A matrix is non singular then after elimination there are no non-zero rows & $|A| = \prod_{j=0}^{n} (d_j)$ where d_j is the diagonal element a_{jj} for upper triangular (U) corresponding to A.

---

pivot $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow \begin{vmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{vmatrix}$

$R_2 \rightarrow R_2 - \frac{c}{a} R_1$

* Now it is in upper triangular form

$\Rightarrow |A| = d_1 \cdot d_2 = a\left(d - \frac{cb}{a}\right)$

Determinant of 2x2 from properties : $|A| = ad - bc$

---

⑨ → $\boxed{\det AB = \det(A) \det(B)}$

$\Rightarrow \det(A^{-1})$

$A^{-1} A = I$

$\Rightarrow \det(A^{-1} A) = \det(I)$

$\Rightarrow \det(A^{-1}) \det(A) = 1$

$\Rightarrow \boxed{\det(A)^{-1} = \det(A^{-1})}$

$\det(A^2) = \det^2(A)$

$\det(2A) = 2^n \det A$

(10) $\det(A^T) = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$ad - bc = ad - cb$$

Determinant is well defined by properties 1-3

Proof #10 Using (1-9)

$$|A^T| \stackrel{?}{=} |A|$$

$$|U^T L^T| \stackrel{?}{=} |LU|$$

$$|U^T|\ |L^T| \stackrel{?}{=} L\ U$$

But they are all triangular
$\Rightarrow$ determinate is product of diagonals
$\Rightarrow$ Diagonal elements stay same for Transpose
$\Rightarrow |U^T| = |U|$ & $|L^T| = |L|$ Hence proved

Determinant formula from properties

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \boxed{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc$$

$$3\times3 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

9 pieces

$$\begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} + \cdots$$

27 pieces

$$\begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} + \cdots$$

- ultimately the final pieces have only 1 element per row.
- Some of them becomes zero
- All the ones with one entry per row & one entry per column

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{vmatrix}$$

$$= +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}[a_{22}a_{33} - a_{23}a_{32}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

BIG FORMULA

$$\det A = \underbrace{\sum \pm}_{\boxed{n!\ \text{terms}}} a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$$

Element in first row can be choosen n ways
Element in second row can be choosen (n-1) ways
⋮
Element in last row cannot be choosen. 1

$(\alpha, \beta, \ldots \omega)$ = Permutation of $(1, 2 \ldots n)$ terms.

Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$a_{14}a_{23}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} = 0$$

$$R_1 - R_2 + R_3 - R_4 \longrightarrow 0$$

$$C_1 + C_3 - C_2 - C_4 \longrightarrow 0$$

Cofactors 3x3

$$\text{determinant} = a_{11}[a_{22}a_{33} - a_{23}a_{32}] + a_{12}[a_{23}a_{31} - a_{21}a_{33}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

Cofactors are in the brackets. ☆

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactors

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} * (-) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactors (- included)

$$\boxed{\text{Cofactor of } a_{ij} = C_{ij} = (-1)^{i+j} * \det\left(\begin{array}{l}n-1 \text{ square matrix} \\ \text{with row } i, \text{ column } j \text{ removed}\end{array}\right)}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}_{3\times3}$$

* Without the build in sign it's called a minor
with built in sign it's called cofactor.

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} \cdots + a_{1n}C_{1n}}$$

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

Example

tridiagonal matrix.

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \end{vmatrix}$$

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$A_1 = |1| = 1$$

$$A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$A_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$|A_4| = a_{11}C_{11} + a_{22}C_{22}$$
$$= a_{11}|A_3| + a_{22}|A_2|$$
$$= |A_3| = \underline{-1}$$

* Tridiagonal matrix is a matrix with non zero elements on the main diagonal, the first diagonal below this & above this

Formula for A^{-1}

2×2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

$C^T \rightarrow \Sigma$ product of $n-1$ entries

$\det A \rightarrow \Sigma$ product of n entries

3×3

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det A} C_A^T$$

$C_A^T \rightarrow$ product of 2 terms

$\det A \rightarrow \Sigma$ product of 3 terms

---

checking $AA^{-1} \stackrel{?}{=} I$

$\Rightarrow A C^T \stackrel{?}{=} (\det A) I$

$$\underset{[A]}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}} \underset{[C^T]}{\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & \cdots & & c_{n2} \\ \vdots & & & \\ c_{1n} & \cdots & & c_{nn} \end{bmatrix}} = \begin{bmatrix} \det A & & 0 & \\ & \det A & & \\ 0 & & \ddots & \\ & & & \det A \end{bmatrix} = \det(A) I$$

$\boxed{a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} + \cdots a_{1n}c_{1n}}$ $= \det A$

Reason

$R_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ C_{R2}

$R_1 C_{R2} = a(-b) + ba$
$= -(ab - ba) = 0$

R_1 times $C_{1st\,row}$ = determinant A

R_n times $C_{m\,row} = 0$
$n \neq m$

let $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ } Finding determinant = R_1 time C_{R1}

Singular matrix

$$= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix}$$

$$= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} b \\ -a \end{bmatrix} = 0$$

This is exactly what is happening in $R_1 C_{R2}$ case but instead of C_{R2} there we have C_{R1} here and the calculated to be same.

* It's as if we taking the determinat of A screwed matrix A_s instead of A
(ie $\det(A_s)$ instead of $\det(A)$ where A_s = matrix A with $R_m = R_n$

* This is what happens when we calculate $\begin{bmatrix} R_n\ C_{Rm} \\ m \neq n \end{bmatrix} = \det A_s = 0$

$A_s \,//\, R_m = R_n$

eg. $R_1 C_{Rn} \Rightarrow 0$
because A_s first & last rows identical $\Rightarrow$ A_s non invertible

---

System $Ax = b$
solution $x = A^{-1}b = \frac{1}{\det A} C^T b$ (without doing elemination) solving.

CRAMER'S RULE

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

for $A_{m \times n}$

$x_1 = \frac{\det B_1}{\det A}$ $\det B_1 = [R_1 \text{ of } C^T] \text{ time } b$

$x_2 = \frac{\det B_2}{\det A}$ $\det B_2 = [R_2 \text{ of } C^T] \text{ times } b$

Column 1 replaced by b

$B_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & \vdots & \vdots & & \vdots \\ \vdots & & & & \\ b_m & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

det of B_1 = [Row 1 of C^T] time b $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Row 1 of $C^T = [C_{11}\ C_{21}\ C_{31} \cdots C_{m1}]$

det of $B_1 = C_{11}b_1 + C_{21}b_2 + C_{31}b_3 + \cdots C_{m1} b_m$

det of B_1 = cofactors along first column weighted by elements of first column

det of B_2 = cofactors along second column weighted by elements of second column.

$B_2 = \begin{bmatrix} a_{11} & & a_{13} & \cdots & a_{1n} \\ a_{21} & & a_{23} & & \vdots \\ \vdots & & \vdots & & \vdots \\ a_{m1} & & a_{m3} & & a_{mn} \end{bmatrix}$

b

Here moved

$\Rightarrow B_j$ = A with column j replaced by column matrix b.

~~$\det B_n = b_1 C_{1n} + b_2 C_{2n} + b_3 C_{3n}$~~

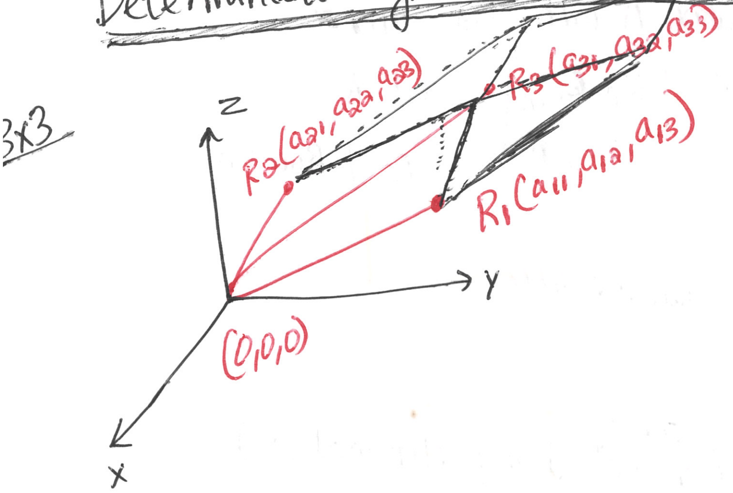
$\det B_j = b_1 C_{1j} + b_2 C_{2j} + \cdots b_m C_{mj}$

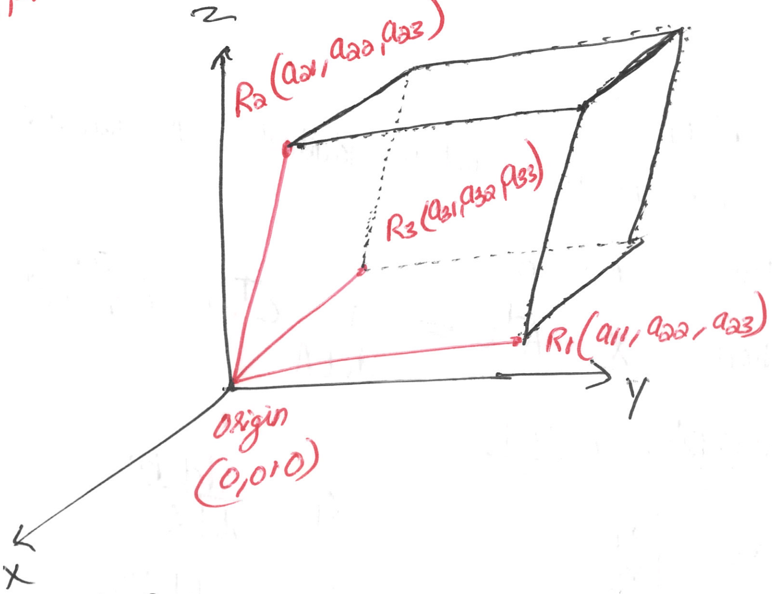
$\Rightarrow x_j = \frac{\det B_j}{\det A}$

* Computationally Cramers Rule is a disaster
* Elimination is the way to go.

Determinant gives a volume in n dimention for [A n×n]

3x3





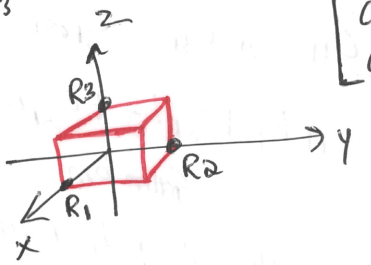
$|\det A|$ = Volume of box defined by $R_1R_2R_3$

sign specifies the cyclic order of R_1, R_2, R_3

---

eg. $A = I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|\det A| = 1$



Property 1

Property (3a)

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eg 2. Orthogonal matrix

$A = Q$

Q = Columns where perpendicular unit vectors

($\det A = \det A^T$ therefore row, column doesnt matter)

$\det Q = \pm 1$

unit cube turned in space.

$Q^TQ = I$

$\det(Q^TQ) = \det I$

$\det(Q^T)\det Q = 1$ [Rule 9]

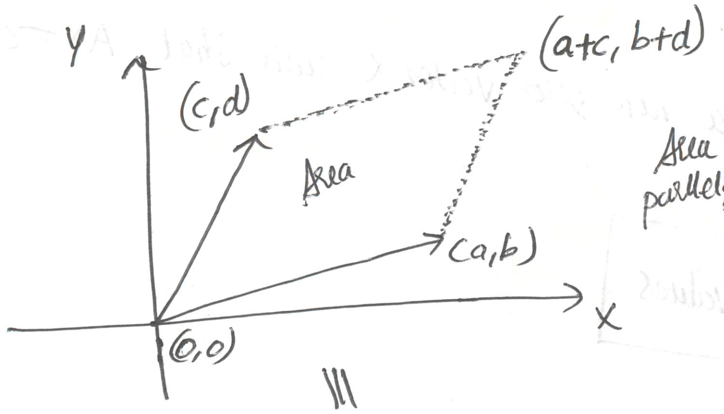
$[\det Q]^2 = 1$ [Rule 10]

$\det Q = \pm 1$ Hence proved.

$\left|\det A\right|$ = Volume of box has properties 1, 2, 3a, I, ±, t

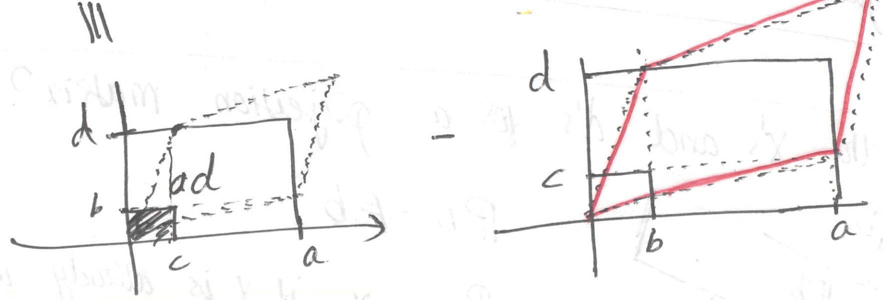
to show 3b

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

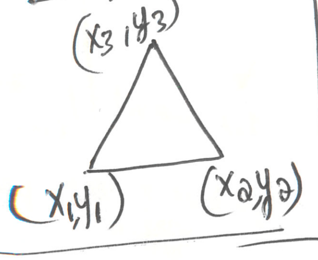


Area parallelogram $= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Area triangle $= \frac{1}{2}(ad-bc)$



triangle



Area of triangle = det of $\frac{1}{2}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}$

$$= \frac{(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)}{2}$$

$$= \frac{ad-bc}{2}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_2 - R \end{matrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left[(x_2-x_1)(y_2-y_1) - (y_2-y_1)(x_3\right.$$

Eigenvalues - Eigenvectors

A matrix square. Ax parallel to x are eigen ~~vecto~~ vectors

$\boxed{x \text{ such that } Ax = \lambda x = \text{eigen vectors of } A}$

★★★ x non zero.

if $\lambda = 0$ $x \in$ Null space $Ax = 0$

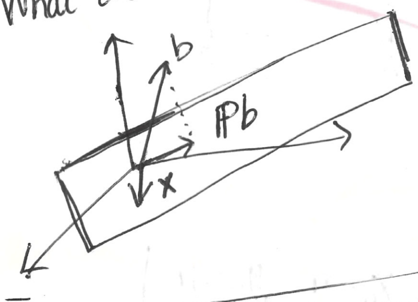
If A is singular then there is a non zero vector x such that $Ax = 0$

$\Rightarrow \lambda = 0$ is a eigen value

* $\boxed{A \text{ singular} \Rightarrow \lambda \text{ is eigen values}}$

---

eg:- What are the x's and λ's for a projection matrix?



$Pb \neq b$

$Px = x$ if x is already in the plane for projection.

* $\boxed{x \text{ is an eigen vector with } \lambda = 1 \text{ for } x \text{ in plane}}$ x in column space

* $\boxed{x \text{ is an eigen vector with } \lambda = 0 \text{ for } x \perp \text{ to plane}}$ x in nullspace (A^T)

---

eg:- Permutation matrix

$Ax = \lambda x$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;	$\lambda = 1$	$Ax = x$
$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;	$\lambda = -1$	$Ax = -x$

* n×n matrix will have n eigen values

* Sum of eigen values = trace of Matrix

Q) How to solve $Ax = \lambda x$; where $X \neq 0$?

$[A - \lambda I] X = 0$ ⟹ $\det(A - \lambda I) = 0$ ⟹ $|A - \lambda I| = 0$

characteristic equation OR Eigen value equation

if solvable for $X = 0$ then $[A - \lambda I]$ must be singular

because if not singular then exist inverse & $[A - \lambda I]^{-1} [A - \lambda I] X = 0$ ⟹ $X = 0$ so singular contradicts

1) Find λ_1 through λ_n from characteristic equation
2) Find eigen vector X from $[A - \lambda I] X = 0$ by elimination

Q) Example $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

STEP I

~~det~~ $|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$ ⟹ $(3-\lambda)^2 - 1 = 0$

$(3 - \lambda) = \pm 1$

$3 \pm 1 = \lambda$

$\lambda = 2, 4$

$\lambda_1 = 2$

$\lambda_2 = 4$

In 2x2 case

$$\boxed{\lambda^2 - (\text{trace})\lambda + |\text{determinant}|}$$

trace = sum of eigen values

|determinant| = product of eigen values

general for n order matrix

STEP II

$\lambda_2 = 4$

$[A - 4I] X = 0$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} X = 0$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\quad X = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 2$

$[A - 2I] X = 0$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X = 0$

$X = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

---

if, $Ax = \lambda x$ — (1)

then $(A + 3I) X = \lambda X + 3X$

$Ax + 3IX = (\lambda + 3)x$ since (1)

* $A + 3I \rightarrow \lambda + 3$

if, $Ax = \lambda x$ $\quad \lambda = \lambda_1, \lambda_2 \dots \lambda_n$

and $Bx = \alpha x$ $\quad \alpha = \alpha_1, \alpha_2 \dots \alpha_n$

then $[A+B]x = (\lambda+\alpha)x$ is wrong. because the x's need not be same for both A & B. } A & B might have different eigen vectors.

in fact $Ax = \lambda x$, $By = \alpha y$ } A+B is futile Caution

---

example. $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

90° rotation

determinant $\lambda_1 \lambda_2 = 1$ } $\Rightarrow \lambda_1(-\lambda_1) = 1$

trace $\lambda_1 + \lambda_2 = 0$ $\quad \lambda_1^2 = -1$ also $\lambda_2^2 = -1$

OR $|Q - \lambda I| = 0 = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm j$

★★★★ In general for ~~[illegible]~~ θ degree rotation of vector x in 2-D then the rotation matrix Q is given by ~~[illegible]~~ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

A always orthogonal

Ax gives the rotated vector

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad$ If triangular λ = diagonal elements.

determinant = product of diagonal

product of eigen value = product of diagonal

sum of eigen value = trace = sum of diagonal

$\Rightarrow$ eigen values = diagonal elements

$|A - \lambda I| = 0$

$\Rightarrow (3-\lambda)^2 = 0$

$\Rightarrow \lambda = 3, 3$

$[A - 3I]x = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$

$x = \begin{bmatrix} k \\ 0 \end{bmatrix}$

* There is only 1 independent eigen vector

- Suppose we have n-linearly independent eigen vectors of A
- Put them in the columns of matrix S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$A = \cancel{\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}}$$

$$AS = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & \\ & \vdots & & \lambda_n \end{bmatrix}$$

* If we have n-independent columns then S is invertible.

* Assuming A has n independent eigen vectors.

$$\boxed{AS = S\Lambda}$$

* where Λ is the eigen value matrix in diagonal position

$\Rightarrow S^{-1}AS = \Lambda$

DIAGONALISATION

$$\boxed{\begin{aligned} S\Lambda S^{-1} &= A \\ S^{-1}AS &= \Lambda \end{aligned}}$$

If $Ax = \lambda x$

then $A^2 x = \lambda A x$

$$\boxed{A^2 x = \lambda^2 x}$$

$A^2 \xrightarrow{\text{eigen value}} \lambda^2$ eigen vectors same.

ALITER

$A^2 = (S\Lambda S^{-1})\, S\Lambda S^{-1}$

$A^2 = S\Lambda (S^{-1}S) \Lambda S^{-1}$

$$\boxed{A^2 = S\Lambda^2 S^{-1}}$$

$S \to$ same, $\Lambda \to \Lambda^2$ } eigen vectors same, $\lambda \to \lambda^2$

In general

$A^k = S\Lambda^k S^{-1}$

eigen values $\lambda \to \lambda^k$

eigen vectors same!

*****

$A = S\Lambda S^{-1}$ is a suitable factorisation for the other factorisations

$A = LU$ (elimination) or $A = QR$ (Gram-Schmidt) especially when powers & polynomials of A is involved

Theorm

$A^k \to 0$ as $k \to 0$ if all $|\lambda_i| < 1$ {provided A has n eigen-vectors otherwise we cannot conclude anything

$A^k = S\Lambda^k S^{-1}$

as $k \to \infty$ $S R S^{-1}$ are not moving.

---

* A is sure to have n independent eigen vectors (and be diagonalisable if all the λ's are different)

* Reapted eigen values $\Longrightarrow$ May or Maynot have 'n' independent eigen vectors.

eigen values } each distinct gives ~~atleast~~ Exactly 1 eigen vector
each repeated given atleast 1 eigen vectors

Suppose A is triangular

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\qquad (A-\lambda I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

eigen values are 2, 2 $\qquad$ eigen $x = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

* Algebraic multiplicity of a eigen value = No. of times repeated
geometric multiplicity of a eigen value = No. of independent eigen vectors.

MULTIPLICTY ≥ 1
ALGEBRIAC MULTIPLICITY $\leq$ GEOMETRIC MULTIPLICTY
$1 \leq$ GEOMETRIC MULTIPLICITY

Equation $u_{k+1} = Au_k$

Start with a given vector u_0

$u_1 = Au_0$
$u_2 = A^2 u_0$
$u_n = A^n u_0$

To really solve: Write u_0 as a combination of eign vectors.

$$u_0 = c_1x_1 + c_2x_2 + \dots c_nx_n$$

$$Au_0 = c_1\lambda x_1 + c_2\lambda_2x_2 + c_3\lambda_3x_3 + \dots + c_n\lambda_nx_n$$

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + c_3\lambda_3^{100}x_3 + \dots + c_n\lambda_n^{100}x_n$$

$$A^{100}u_0 = S\Lambda^{100}c = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^{100} & & 0 \\ & \lambda_2^{100} & \\ 0 & & \ddots \; \lambda_n^{100} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example: Fibonacci $0, 1, 1, 2, 3, 5, 8, 13, 21, 34$ $F_{100} = ?$

$$F_{k+2} = F_{k+1} + F_k$$ — second order difference equation

convert to system of first order equation.

Let $U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$U_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$ → System of simple equations.

$$\begin{bmatrix} U_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U_k \end{bmatrix}$$

$$U_{k+1} = A \cdot U_k$$

$$F_{100} \approx c\left(\frac{1+\sqrt{5}}{2}\right)^{100}$$ because

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 \uparrow + c_2\lambda_2^{100}x_2 \downarrow$$

because $|\lambda_1| > 1$, $|\lambda_2| < 1$

eigen values

$$\lambda^2 - 1\lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -0.618$$

x_1?

x_2?

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

because $x \in [N(A)]$

$$\rightarrow x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_0 = \overset{?}{c_1} x_1 + \overset{?}{c_2} x_2$$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k u_0$$

$$u_k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$u_k = A^k u_0$$

$$\Rightarrow u_k = S \Lambda^k S^{-1} u_0$$

but u_0 can be written as linear combination of eigen vectors.

$$u_0 = Sc$$

$$\rightarrow u_k = S \Lambda^k S^{-1} (Sc)$$

$$\underline{\underline{u_k = S \Lambda^k c}}$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 \lambda_1 + c_2 \lambda_2 = 1$$

$$c_1 + c_2 = 0$$

$$c_1 = -c_2$$

$$\Rightarrow c_1 \lambda_1 - c_1 \lambda_2 = 1$$

$$c_1(\lambda_1 - \lambda_2) = 1$$

$$c_1 = (1.618 + 0.618)^{-1}$$

$$c_2 = -(1.618 + 0.618)^{-1}$$

$$\Rightarrow u_k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ \frac{-1}{\lambda_1 - \lambda_2} \end{bmatrix}$$

<u>Example</u> Solving Linear Differential Equations with Constant Coefficients.

$\longrightarrow$ System of First Order, First derivative, constant coefficient, Linear Equations.

$$\left.\begin{aligned} \frac{du_1}{dt} &= -u_1 + 2u_2 \\ \frac{du_2}{dt} &= u_1 - 2u_2 \end{aligned}\right\} \quad A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\dot{u}(t) = A\,u(t) \longrightarrow \text{system}$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\lambda^2 + 3\lambda = 0$$
$$\lambda(\lambda+3) = 0$$
$$\lambda_1 = 0$$
$$\lambda_2 = -3$$

$$[A - \lambda_1 I]x = 0$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = 0$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[A - \lambda_2 I]\, x = 0$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = 0 \Rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

check

$$\frac{du}{dt} = Au \quad (\text{Plug in } e^{\lambda_1 t} x_1)$$

$$\lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$$

$\lambda x_1 = A x_1$ Hence proved!

$$u_{k+1} = A u_k$$

solution

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots$$

1st order linear difference equation.

$$\dot{u} = A u \qquad \dot{u}(t) = A\, u(t)$$

solution

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + \dots$$

1st order Linear differential equation

Using initial condition for c_1 & c_2

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 e^{0t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-30}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$2c_1 - c_2 = 1$$
$$c_1 + c_2 = 0$$
$$c_1 = -c_2$$
$$3c_1 = 1$$
$$\Rightarrow c_1 = 1/3$$
$$c_2 = -1/3$$

$$u(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left(\frac{-1}{3}\right) e^{-3t}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

$$u(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{3} e^{-3t}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

$$u(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

① Stability $u(t) \to 0$ as $t \to \infty$ / need $e^{\lambda t} \to 0$ as $t \to \infty$ / ~~$\lambda < 0$~~ Real$[\lambda] < 0$

② Steady state $\lambda_1 = 0$ and others eigen values $Re(\lambda) < 0$

③ Blow up if any $Re(\lambda) > 0$

* For 2x2 matrix

① Stability

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

trace $= a + d = \lambda_1 + \lambda_2$

$\Rightarrow a + d < 0$

trace -ve

determinant +ve

initial conditions

$u(0) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$u(0) = Sc$ S - matrix for eigen vectors

$\Rightarrow Sc = u_0$

solve for c

express the initial setup as a linear combination of the eigen vectors

General solution in terms of S & λ

$\frac{du}{dt} = Au$ but $A = S \Lambda S^{-1}$

$\cancel{\frac{du}{dt} = S\Lambda S u}$ $\frac{du}{dt} = S \Lambda S^{-1} u$

Set $S^{-1} u = v$

$u = Sv$

$\frac{du}{dt} = S \frac{dv}{dt}$

$S^{-1} \frac{du}{dt} = v$

$\cancel{S \frac{du}{dt} = A S u} \Rightarrow S^{-1} \frac{du}{dt} = \Lambda v$

$\Rightarrow \boxed{\frac{dv}{dt} = \Lambda v}$

→ diagonalising A

→ Uncoupling

$\begin{matrix} \frac{dv_1}{dt} = \\ \frac{dv_2}{dt} = \end{matrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{matrix} \frac{dv_1}{dt} = \lambda_1 v_1 \\ \frac{dv_2}{dt} = \lambda_2 v_2 \end{matrix}$

$V_1 = c_1 e^{\lambda_1 t}$
$V_2 = c_2 e^{\lambda_2 t}$
$\Rightarrow V(t) = e^{\Lambda t} V(0)$

$c_1 = V_1(0)$
$c_2 = V_2(0)$

$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{A(t)} u(0)$

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$$

Matrix exponential (using power series)

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots \frac{(At)^n}{n!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)$$

$$(1-x)^{-1} = \sum_{n=0}^{\infty} (x^n)$$

$$(I - At)^{-1} = I + (At) + (At)^2 + (At)^3 + \dots$$

= given |Eigen values (At)| < 1

put $A = S\Lambda S^{-1}$

$$e^{At} = I + (S\Lambda S^{-1} t) + \frac{(S\Lambda S^{-1} t)^2}{2!} + \frac{(S\Lambda S^{-1} t)^3}{3!}$$

$$e^{At} = I + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1} t^2}{2!} + \dots$$

(since t is scalar)

$$e^{At} = I + S\left(\frac{\Lambda t}{1!} + \frac{\Lambda^2 t^2}{2!} + \frac{\Lambda^3 t^3}{3!} + \dots\right) S^{-1}$$

$$e^{At} = S\left(I + \frac{\Lambda t}{1!} + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \dots\right) S^{-1}$$

since $S I S^{-1} = S S^{-1} = I$.

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$$ proved

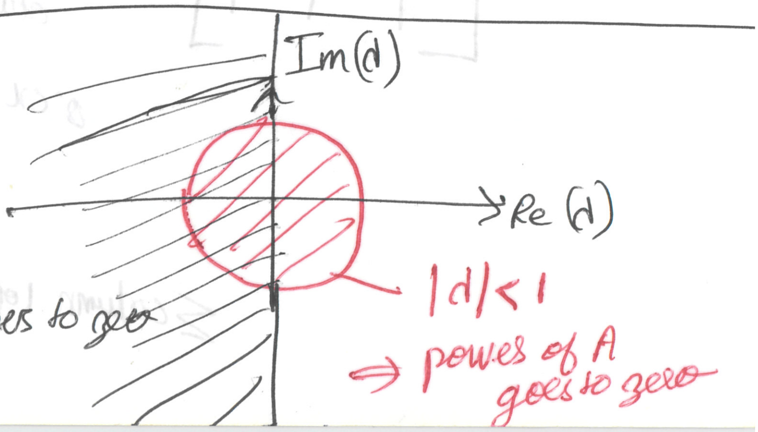
provided A can be diagonilised
$\Rightarrow S^{-1}$ exist
$\Rightarrow$ Independ eigen vectors exists.

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

~~$e^{\lambda t} \to 0$ as $t \to 0$~~
~~if $\lambda_i \; \forall i \in (1,n)$ less than zero~~

$e^{\Lambda t} \to 0$ as $t \to 0$
if $Re(\lambda_i) < 0 \; \forall i \in (1,n)$

$\Rightarrow$ Exponential of A goes to zero



$Re|\lambda_i| < 0 \rightarrow$ exponential of A: $e^{At} \rightarrow 0$ as $t \rightarrow 0$

$|\lambda| < 1 \Rightarrow$ powers of A: $A^n \rightarrow 0$ as $n \rightarrow 0$

$y'' + by' + ky = 0 \longrightarrow$ Change 1 2nd order into 2,2 1st order system

let $u = \begin{bmatrix} y' \\ y \end{bmatrix}$

let $\dot{u} = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

$A = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \dot{u} = Au$

General case

$A = \begin{bmatrix} - & - & - & - \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4th order to 4x4 1st order

$A = \begin{bmatrix} 0.1 & 0.1 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$

Markov Matrix

(1) All matrix entries > 0 } Hold for powers also

(2) All columns add to 1

(3) Powers of Markov Matrix is another matrix

A can be diagonalised

$\Rightarrow$ S^{-1} exist

$\Leftarrow$ Independent eigen vectors exists

$|\lambda| < 1$

$\Leftarrow$ powers of A goes to zero

$A^2 = \begin{bmatrix} | & | & | \\ | & | & | \end{bmatrix}$ = linear combination of columns of A linear combination coefficients also add to 1

e.g. column 1 of A^2

$= 0.1 \begin{bmatrix} Ac_1 \end{bmatrix} + \begin{bmatrix} 0.2 \end{bmatrix} \begin{bmatrix} Ac_2 \end{bmatrix}^2 + 0.7 \begin{bmatrix} Ac_3 \end{bmatrix}^2$

Σ column 1 of A^2 $= 0.1 \times 1 + 0.2 \times 1 + 0.7 \times 1$

$= \Sigma$ column 1 of $A = 1$

Properties of Markov Matrix

- Eigen value of ① $\lambda = 1$
- All other eigen values $|\lambda_i| \leq 1$ — exceptional case.
- $u_k = A^k u_0 = x_1 c_1 \lambda_1^k + x_2 c_2 \lambda_2^k + \text{-----}$
- $u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$

  $\lambda_1^k \to 1$

  all others small than one tends to zero as $k \to \infty$

- $u_\infty = c_1 x_1$
- eigen vector X_1 corresponding to $\lambda_1 = 1$ has all components +ve

---

Q) How if colums add to one $\lambda = 1$ is eigen value?

show $(A - I) = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & 0.6 \end{bmatrix}$

$\downarrow$

$|A - \lambda I| = 0$

show $|A - I| = 0$

or $(A - I)$ is singular

$\Rightarrow$ 1 is a eigen value

$\Rightarrow$ All columns now add to zero

$\Rightarrow$ $(A - I)$ is singular

$\Rightarrow$ $[1, 1, 1] \in \cancel{N(A)}\ N(A - I)^T$

$\Rightarrow$ $X_1 \in \cancel{N(A)}\ N(A - I)$

$\boxed{\text{eigen Values of } A = \text{eigen Values } A^T}$

proof

$\det(A - \lambda I) = 0$

$\det(A - \lambda I)^T = 0$

$\det(A^T - \lambda I) = 0$

$\Rightarrow \lambda \text{ for } A = \lambda \text{ for } A^T$

$$\begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix} \begin{bmatrix} 0.6 \\ 33 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} 0.6 \\ 33 \\ 0.7 \end{bmatrix}$$

Application of Markov matrices

$u_{k+1} = A u_k$ where A is markov

$$\begin{bmatrix} u_{cal} \\ u_{mas} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mas} \end{bmatrix}_k$$

Steady state

Starting $\begin{bmatrix} u_{cal} \\ u_{mas} \end{bmatrix}_{k=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

$$u_k = A^k u_0$$

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

$\lambda_1 = 1$ (from Markov property)

$\lambda_2 = 0.7$ (from trace $= \lambda_1 + \lambda_2$)

$$(A - I) = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} X_1 \end{bmatrix} = 0$$

$$\Rightarrow X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$$

as $k \to \infty$ $\quad u_\infty = c_1 x_1 = \begin{bmatrix} 2c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 666.66 \\ 333.33 \end{bmatrix}$

$$(A - 0.7I) = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = 0 \qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 \lambda_1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} 0.7^k$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (0.7)^k \cdot \frac{2000}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Projections with orthonormal Basis $(q_1, q_2 \cdots q_n)$ in n dimensional space

√ expansion.

any $V = x_1 q_1 + x_2 q_2 + \cdots x_n q_n$

How to get x_1 or any x_n — |projection of V on q_n|

ie |projection of V on q_1|

$$V = x_1 q_1 + x_2 q_2 + \text{----} \; x_n q_n$$

$$\boxed{q_1^T \cdot V = x_1}$$ [since $q_1 \cdots q_n$] = orthonormal Basis.

$$V = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = V$$

in matrix language

$$V = Qx$$

$$x = Q^{-1} V$$

$$\boxed{x = Q^T V}$$ [since Q is orthonormal matrix]

Fourier Series

$$f(x) = \underbrace{a_0}_{q_1} + \underbrace{a_1 \cos x}_{q_2} + b_1 \underbrace{\sin x}_{q_3} + a_2 \underbrace{\cos 2x}_{q_4} + b_2 \sin 2x$$

[infinite dimensional vector space]

$(q_1 \cdots q_\infty)$ are orthogonal

dot product of function

For vectors

$$V^T W = V_1 W_1 + V_n W_n$$

but for functions $f(x)$ & $g(x)$

$$f^T g = \int_{-\infty}^{\infty} f(x) g(x) \cdot dx =$$

orthogonal vectors $(V^T W = 0)$

orthogonal functions.

$$f^T g = 0 \quad \int_{-\infty}^{\infty} f(x) \cdot g(x) \cdot dx = 0$$

How to get a_1

$$\int_0^{2\pi} f(x) \cos(x)\, dx = a_1 \int_0^{2\pi} (\cos x)^2 \cdot dx$$

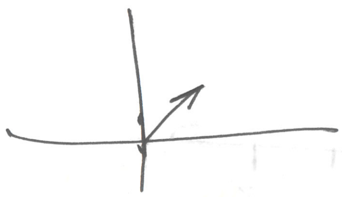
$$\Rightarrow \boxed{a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(x) \cdot dx}$$

Review

- Orthogonality, Orthonormal matrices
- gram-Schmidt
- det A, properties of determinants
- Big formula for determinant — Cofactor formula
- Formula for A^{-1}
- Eigen values & Eigen Vectors
- Diagonalisation
- Differential & Difference equations

1a) $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad P$? that projects b onto a

$$P = A(A^TA)^{-1}A^T \qquad A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = a$$



$$P = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

1b) Eigen values of P?

column space of P = line through a

eigen values $= 0, 0, 1 \rightarrow$ trace. or $Pa = a \Rightarrow \lambda = 1$

singular & $N(A)$ = 2dimensional

1c) Solve $u_{k+1} = P u_k$ $\quad u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$ & Find u_k.

$$u_k = P^k u_0$$

$$u_1 = P u_0$$

$$u_1 = \frac{aa^T}{a^Ta} u_0 = a\left[\frac{27}{9}\right] = 3a = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$u_k = P^k u_0 = P u_0 \quad \left[\text{since } P^k = P\right] \; \forall k \geq 1$$

or treating P as any other matrix

then. $u_k = A^k u_0 = c_1 d_1^k x_1 + c_2 d_2^k x_2 + c_3 d_3^k x_3$

$U_k = c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ for $k>0$ $\quad d_2, d_3 = 0$

$d_1 = 1$

since $0^k = 0 \; \forall k > 0$

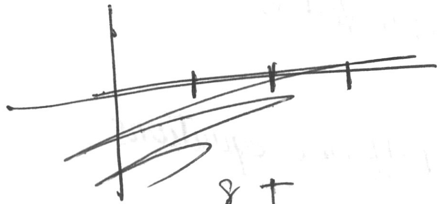
Q2a) $t=1$ $y=4$

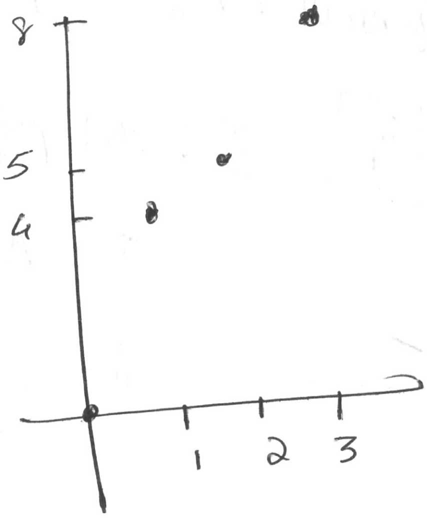
$t=2$ $y=5$

$t=3$ $y=8$

$\boxed{t=0 \quad y=0}$

mandatory!





let $y = Dt$

$\left.\begin{array}{l} 1 \cdot D = 4 \\ 2 \cdot D = 5 \\ 3D = 8 \end{array}\right\}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$

$A\hat{x} = b$

best fit: $(A^T A \hat{x} = A^T b)$

$A^T A = 14 \Rightarrow 14\hat{D} = 38$

$\hat{D} = \frac{38}{14} = \frac{19}{7}$

* $\hat{x}$ = projecting b on to column space of A

2b) If 2 vectors $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ find orthogonal ~~vectors~~ basis using gram-smith

$A = a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$B \perp A$ - $B = a_2 - (a_2 \text{ projection on } A)$

$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{a a_1^T}{a_1^T a_1}\right) a_1$

$$B = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{6}{104}\begin{bmatrix}1\\2\\3\end{bmatrix} =$$

3.) 4x4 matrix Eigen value $\lambda_1 \lambda_2 \lambda_3 \lambda_4$

a) Invertib? when Ans) $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \neq 0$

b) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{\lambda_1 \lambda_2 \lambda_3 \lambda_4}$

c) trace of $(A+I) = 4 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$

---

tridiagonal family. Use cofactors to show that

4) $A_{4\times4} = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 1 & 1 & 0\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}$ $D_n = \_\_ D_{n-1} + \_\_ D_{(n-2)}$

Cofactors along R1

$D_n = \det(A_n)$

$C_{11} = D_{n-1}$

$C_{12} = - \begin{vmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 1 & 1\end{vmatrix}$

$D_n = a_{11}C_{11} + a_{12}C_{12}$

$C_{12} = - D_{(n-2)}$

$D_n = D_{(n-1)} - D_{(n-2)}$

4b) Solve for system D_n

$D_1 = 1$

$D_2 = \begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} = 0$

$$\begin{bmatrix}D_n\\D_{n-1}\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}D_{n-1}\\D_{n-2}\end{bmatrix}$$

$U_{n+1} = [A]\, U_n$ where $A = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$

λ_1, λ_2

$\lambda^2 - \lambda + 1 = 0$

$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}j}{2}$

$\Rightarrow \lambda_1 = \frac{1+i\sqrt{3}}{2}$

$\lambda_2 = \frac{1-i\sqrt{3}}{2}$

$|\lambda_1| = 1$

$|\lambda_2| = 1$

$\lambda_1 = e^{i\pi/3}$

$\lambda_2 = e^{-i\pi/3}$

---

5) $A_4 =$ Family $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ Find the $\mathbb{P}$ to the column space of A^3.

$A^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ $\mathbb{P} = A(A^T A)^{-1} A$

5a) Find A^3 λs & $\vec{x}$ eigens.

$|A^3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$ $-\lambda^3 + 5\lambda$

$\lambda(-\lambda^2 + 5)$

$\lambda = 0, \sqrt{5}, -\sqrt{5}$

$\mathbb{P}$ to A_4: column space

$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

R_1 cofactors

$C_{12} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{vmatrix}$

$C_{12} = -\begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} = \underline{\underline{9}}$

$|A_4| = a_{12}C_{12} = \underline{\underline{9}}$

$\therefore$ Column space of (A_4) spans whole $\mathbb{R}^4$.

$\therefore$ $\mathbb{P}$ (A_4 column) $= I$

(Real) Symmetric Matrices

① • Eigen values are also real

② • Eigen vectors are perpendicular [can be choosen] — normally they are independent but not perpendicular

Usual case $A = S\Lambda S^{-1}$

Symmetric case $A = Q\Lambda Q^{-1}$

$A = Q\Lambda Q^{T}$

$A^T = (Q\Lambda Q^T)^T$

$A^T = Q^{T^T}\Lambda^T Q^t$

$A = A^T = Q\Lambda Q^T$

S is scalable to orthonormal matrix in case of symmetric matrix A.

Q - orthonormal

orthonormal eigen vectors = colums of Q

$Q^{-1} = Q^T$ as Q is square

$Q^TQ = I$

$\Rightarrow Q^{-1} = Q^T$

$\boxed{A = Q\Lambda Q^T}$ → Spectrum theorm in Maths

→ Principle axis theorm

Why real eigen values?

$Ax = \lambda x \xrightarrow{\text{always}} A^*x^* = \lambda^* x^*$

$(a+ib)^* = (a-ib)$

for real matrix $\Rightarrow A^* = A$

$\Rightarrow Ax^* = \lambda^* x^* \Rightarrow$ for real matrix λ comes in λ, λ^* pairs

Apply transpose — $x^{*T}A^T = x^{*T}\lambda^*$

λ is not a matrix, but scalar

$A^T = A$ for symmetric Matrix : $x^{*T}A\,x = (\lambda^*)\,x^{*T}x$ — ①

but $Ax = \lambda x$

$x^{*T}Ax = \lambda\, x^{*T}x$ — ②

$\Rightarrow \lambda^* = \lambda$ from (1 & 2)

since $x^{*T}x \neq 0$

$[x_1^* \; x_2^* \cdots x_n^*] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ always +ve

for $x \neq 0$ vector

x cannot be zero vector since eigen vector.

If A had been complex

then to make same thing work $A^{*T} = A$

$Ax = \lambda x \Rightarrow$ $A^* x^* = \lambda^* x^*$ —— conjugate both sides

(λ = eigen value for x = eigen vector $x \neq 0$)

$x^{*T} A^{*T} = \lambda^* x^{*T}$ —— transpose both sides, λ is scalar

$x^{*T} A = \lambda^* x^{*T}$ —— Using $A^{*T} = A$

① —— $x^{*T} A x = \lambda^* x^{*T} x$ —— post-multiplying x

also, $Ax = \lambda x$ —— ~~the~~ what we know

$x^{*T} A x = \lambda x^{*T} x$ —— pre-multiplying by x^{*T}

② ——

① LHS = ② LHS $\Rightarrow$ ① = 2

$\lambda^* x^{*T} x = \lambda x^{*T} x$

$\Rightarrow \lambda^* = \lambda$

$x^{*T} x = [x_1^* \; x_2^* \cdots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1^* x_1 + x_2^* x_2 + \dots x_n^* x_n$

$= |x_1|^2 + |x_2|^2 + \dots |x_n|^2$

$=> 0$ alway for $x \neq 0$ which it is since x is eigen

If A real

$A^* = A$

$\Rightarrow A^{*T} = A$

$\Rightarrow A^T = A$ symmetric

If A complex

$A^{*T} = A$

$\Rightarrow$ ~~skew~~ Hermitian symmetric

---

if A real & $A = A^T$

$A = Q \Lambda Q^T$

$= \begin{bmatrix} q_1 & q_2 & q_3 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$

$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_n q_n q_n^T$

$= \lambda_1 P_1 + \lambda_2 P_2 + \dots \lambda_n P_n$

~~AA~~ $= P = A(A^T A)^{-1} A$

or Row matrix a $\quad P = \frac{a a^T}{a^T a} = \frac{q q^T}{q^T q} = q q^T$

but $q^T q = 1$ since orthogonal

Every symmetric matrix is a combination of perpendicular projection matrix.

since $q_1 q_2 \cdots q_n$ are perpendicular to each other

Another property

Signs of Pivots are same as sign of λ's

$\#$(positive) pivots $= \#$ positive λ's

Positive Definite Matrix [always symmetric
Positive definite $\in$ Symmetric matrix]

- all eigen values are positive
- all pivots are positive

eg:- $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1 \times \frac{2}{5}$

$\begin{bmatrix} 5 & 2 \\ 0 & 3-\frac{4}{5} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 0 & \frac{11}{5} \end{bmatrix}$ pivots $5, \frac{11}{5} > 0$

$$\left|\begin{bmatrix} (5-\lambda) & 2 \\ 2 & (3-\lambda) \end{bmatrix}\right| = (5-\lambda)(3-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-44}}{2} = 4 \pm \sqrt{5}$$

$$\lambda_1 = 4+\sqrt{5} > 0$$
$$\lambda_2 = 4-\sqrt{5}$$

- ~~determinant~~ all sub determinants (Minors) are +ve

Fourier Matrix : F_n n^2 multiplications

but FFT changed it to $n \log n$

Length

z = Vector with complex entries. $\text{length}^2 \neq z^T z$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$ $\boxed{\text{length}^2 = z^{*T} z}$ always > 0

in C_n instead of R_n H *When we transpose we should also conjugate.

---

eg: $z^T = [1 \quad i]$ $z^T z = [1 \quad i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = \underline{0}$

$z = \begin{bmatrix} 1 \\ i \end{bmatrix}$ but $z^{*T} z = [1 \quad -i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = \underline{\underline{2}}$

length $= \sqrt{2}$

$\therefore \text{length}^2 = z^H z = z^{*T} z$

$H = *T$ (Hermitian)

Inner product

$y^T x$ is no good

$y^{*T} x$ is inner product of complex vectors y & x

$= y^H x$

Symmetric Matrix // Hermitian Matrix

$A^T = A$ is no good for A complex

$A^{*T} = A \Rightarrow A^H A$

$\boxed{\text{Hermitian Matrix : } A^H = A}$

eg:- $\begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix}$

- Real eigen Values
- Perpendicular eigen Vectors
- Similar to symmetric Matrix

Perpendicular eigen Vectors $q_1\ q_2 \cdots q_n$
are perpendicular

$$\text{if} \quad q_i^H q_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Real

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$ is ~~orthonormal~~ orthogonal if $Q^T Q = I$

complex

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$ is Unitary if $Q^H Q = I$

if Q is square then $Q^{-1} = Q^H$

or $Q^{-1} = Q^{*T}$

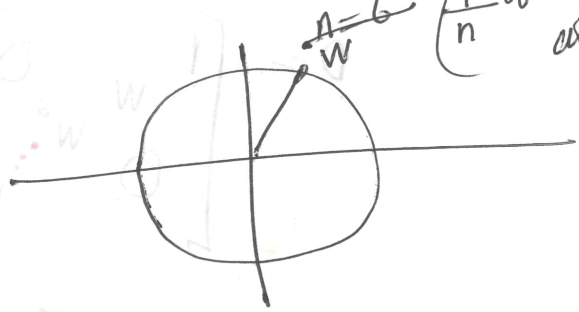
Fourier Matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{bmatrix}$$

$(F_n)_{ij} = w^{ij}$ $\quad j, i \in (0, 1, \ldots n-1)$ $\forall$ n, n matrix (F_n)

starts with 0 ... ends at $(n-1)$ = Engineers notation.

$$\left.\begin{aligned} w^n &= 1 \\ w &= e^{i2\pi/n} \\ w^n &= e^{i2\pi} \\ w &= \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \end{aligned}\right\} n = \text{order of matrix } (F_n)$$



$\left(\frac{1}{n}^{th} \text{ of the way around}\right)$

for $n=4$

$$\left.\begin{aligned} w &= i \\ w^2 &= -1 \\ w^3 &= -i \\ w^4 &= 1 \end{aligned}\right\} (F_4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

* Columns are orthogonal.
* Columns are not orthonormal
* But can be scaled to be orthonormal

$F_4^H F_4 = I \longrightarrow$ after scalling F_4 by length

$\Rightarrow F_4^{-1} = F_4^H$

What property leads ~~to~~ to FFT?

F(64) is connected to F(32)

W in F(64) is 1/2 the way around than W in F(32)

$$\left.\begin{array}{l} W_{64} = e^{i\frac{2\pi}{64}} \\ W_{32} = e^{i\frac{2\pi}{32}} \end{array}\right\}$$ Since this

F_{64} is connected to F_{32}

$(W_{64})^2 = W_{32}$

$$F_{64} \neq \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \rightarrow (32)^2 \times 2 \text{ Calculations.}$$

64^2 calculation

$$F_{64} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \ddots \\ & 1 & & 1 \end{bmatrix}$$

$(64)^2$

$2 \times (32)^2 + 32$

$$D = \begin{bmatrix} 1 & & & 0 \\ & W & & \\ & & W^2 & \\ 0 & & & \ddots \; W^{31} \end{bmatrix}$$

Now break down $F_{32} \longrightarrow F_{16}$

$$F_{64} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \left[\begin{array}{cc|cc} F_{16} & 0 & & \\ 0 & F_{16} & & 0 \\ \hline & & F_{16} & 0 \\ & 0 & 0 & F_{16} \end{array}\right] \left[\begin{array}{c|c} P_{32} & \\ \hline & P_{32} \end{array}\right] \begin{bmatrix} P_{64} \end{bmatrix}$$

$$\updownarrow$$

$$\left[\begin{array}{cc|cc} I & +D & & \\ I & -D & & 0 \\ \hline & & I & D \\ & 0 & I & -D \end{array}\right] *$$

$$\text{Cost} = 2\big[\underbrace{2[16^2]+16}_{(32^2) \text{ is replaced}}\big]+32$$

if we do this recursely.

$$2\big[2[2(8^2)+8]+16\big]+32$$

$$2\big[2[2[2\,\boxed{(4)^2}+4]+8]+16\big]+32$$

$32+32+32+\cdots$ $n \log n$ times

$$\text{Cost} = 6 \times 32 = \frac{1}{2} n \log_2 n$$

if $n = 2^{10}$, $n = 1024$

older method: $n^2 \approx 1000{,}000 = 1024 \times 1024$

FFT: $\frac{1}{2} n \log_2 n = 1024 \times \frac{1}{2} \times 10 = 5 \times 1024$

} 200 more FFT in the time before we could do 1

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Test 1 Eigen values

① $\lambda_1 > 0$
$\lambda_2 > 0$

Test 2: Subdeterminants

$a > 0$

$ac - b^2 > 0$

Test 3 Pivot

Pivot 1: $a > 0$

Pivot 2: $\dfrac{ac - b^2}{a} > 0$

$$c - b\left[\frac{b}{a}\right] = \frac{ac - b^2}{a} > 0$$

* **Test 4**

$x^T A x > 0$

Examples

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$ is on border line — Positive Semi definite.

$\lambda_1 = 0, 20$

$x^T A x$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = 2x_1^2 + 6x_2x_1 + 6x_1x_2 + 18x_2^2$$

$$= \underset{a}{2}x_1^2 + \underset{2b}{12}x_1x_2 + \underset{c}{18}x_2^2$$

Quadratic form

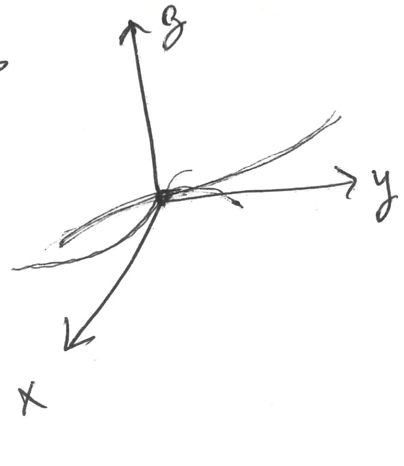
If it positive $\forall$ x_1 & x_2 then A is +ve definite.

$2x_1^2 + 12x_1x_2 + 18x_2^2$

If $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ $\quad x^T A x = 2x_1^2 + 12x_1x_2 + 7x_2^2$

$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x^T A x = 7 - 12 = \underline{\underline{-5}}$

Graph



$z = 2x^2 + 12xy + 7y^2$

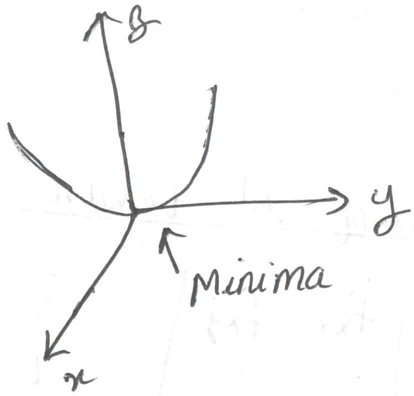
has saddle points.

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \quad x'Ax = 2x_1^2 + 12x_1x_2 + 20x_2^2$$

graph $z = 2x^2 + 12xy + 20y^2$

$|A| = 4$

trace $= 22$



* Positive definite matrices are always symmetric

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

$$x^TAx = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + cx_2 \\ cx_1 + bx_2 \end{bmatrix}$$

$$= ax_1^2 + cx_2x_1 + bx_2^2 + cx_1x_2$$

$$ax_1^2 + 2cx_1x_2 + b^2x_2$$

if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$\boxed{x^TAx = ax_1^2 + 2bx_1x_2 + cx_2^2}$$

- 1'st derivatives are zero at origin.

Calculus

- 2^{nd} derivative +ve } minima.
- 1^{st} derivative 0

18.06 LA

$f(x_1, x_2, x_3 \dots x_n)$

MIN ⇒ Matrix of second derivatives is +ve definite.

- If +ve definite function it can be expressed as a square.

$2(x+3y^2) + 2y^2 = 2x^2 + 12xy + 20y^2$ } Therefore sum of squares > 0

$f(x,y) = 2x^2 + 12xy + 20y^2 = z$

if contant $c = 2(x+3y)^2 + 2y^2$ —— ellipse for +ve definite

if contant $c = 2x^2 + 12xy + 7y^2$ —— hyperbola for saddle point.

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \rightarrow R_2 - R_2 - 3R_1 \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$A = LU$ decompose

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} = LU$$

Multiplies

$$2(x + 3y)^2 + 2y^2$$

outside squares = pivots
inside squares = multipliers.

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$$2(1x+3y)^2 + 2(0x+1y)^2$$

Matrix of 2nd derivative 2x2

$$A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

(r) $f_{xx} > 0$

(t) $f_{yy} > 0$

(s) $f_{xy} = f_{yx}$ — symmetric

$rt - s^2 > 0$ — minima

→ determinant > 0 — minima

A = positive definite for f has minima at origin.

* Same approach extended to nxn.

Example 3x3

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

sub determinant or Minors

$M_{11} = 2$

$M_{11,22} = 3$

$M_{11,33} = 4$

eigen values

$2-\sqrt{2},\ 2,\ 2+\sqrt{2}$

Pivots

$a_{11} = 2$

$a_{22} = 3/2$

$a_{33} = 4/3$

because product of correspond pivots gives determinant

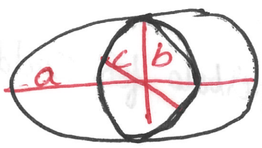
$a_{11} = M_{11}$

$a_{11}a_{22} = M_{11,22} = 2 \times \frac{3}{2} = 3$

$a_{11}a_{22}a_{33} = M_{11,33} = 2 \times \frac{3}{2} \times \frac{4}{3} = 4$

$$x^T A x = 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$$

$f(x,y,z) = f(x_1, x_2, x_3) = x^T A x$ — ellipsoid



$a \neq b \neq c$

major axis a
Middle axis b
minor axis c

} length determined by eigen values.

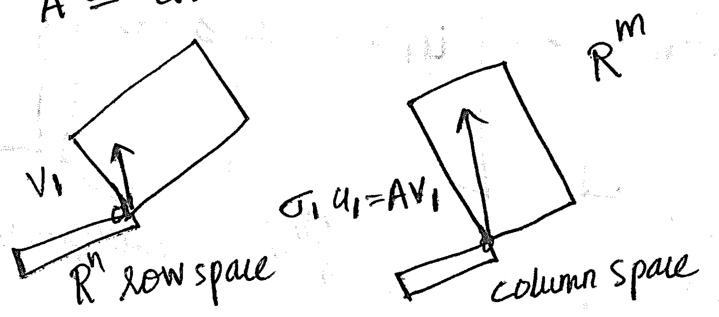
$A = Q \Lambda Q^T$ — Symmetric positive definite.

Singular Value Decomposition (SVD)

* $A = (\text{orthogonal})(\text{Diagonal})(\text{Another orthogonal})$

$A = S \Lambda S^{-1}$ — ordinary if eigen vectors independent

$A = Q \Lambda Q^T$ — symmetric matrix



~~orthogonal~~ normal basis in row space $\{V_1, V_2 \dots V_n\}$ $\Rightarrow$ which is taken to ~~orthogonal~~ normal basis in ~~column space~~ column space $\{u_1, u_2 \dots u_n\}$

~~when~~ where $\sigma_1 u_1 = AV_1$

$\sigma_2 u_2 = A V_2$

$\sigma_3 u_n = A V_n$

if $A = n \times n$

then $\sigma U = AV$

where $U = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}_{n \times n}$ $\quad V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}_{n \times n}$

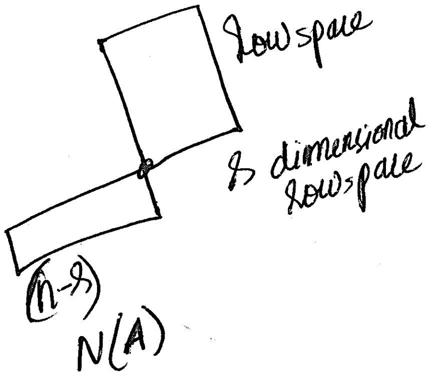
$$\begin{bmatrix} | & | & | & & | \\ u_1 & u_2 & u_3 & \dots & u_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 \ddots & \\ & & & \sigma_n \end{bmatrix} = A \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \dots & v_n \\ | & | & | & & | \end{bmatrix}$$

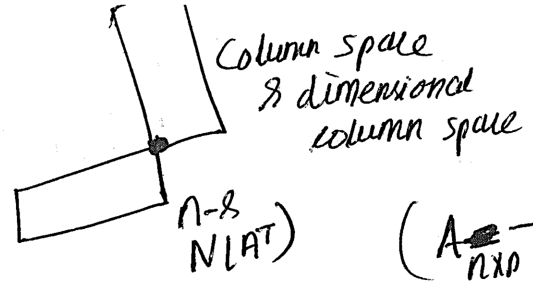
$AV = U\Sigma$

where V, Σ are orthogonal

$\{V_1 \dots V_n\}$ orthonormal

$\{u_1 \dots u_n\}$ orthonormal.





$(A_{n\times n} - \text{square})$

Then
$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r & v_{r+1} \cdots v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} \cdots u_n \end{bmatrix}\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \;\; 0 \end{bmatrix}$$

$v_1 \cdots v_r$: r dim rowspace; $v_{r+1} \cdots v_n$: n-r dim nullspace(A)

$u_1 \cdots u_r$: column space; $u_{r+1} \cdots u_n$: nullspace(A^T)

* So null space don't create problems

Problem $\begin{cases} \text{find } v_1 \cdots v_r \text{ in row space } R(A) \\ \text{find } u_1 \cdots u_r \text{ in colum space } C(A) \end{cases} \sigma_1 \cdots \sigma_r > 0$

$AV = U\Sigma$

$A = U\Sigma V^{-1}$ — inverse of orthogonal matrix is transpose.

$\Rightarrow A = U\Sigma V^T$

$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$

$A^TA = V\Sigma^T(U^TU)\Sigma V^T$

$A^TA = V\Sigma^T\Sigma V^T$

$A^TA = V\Sigma^2 V^T$ (Σ = diagonal)

for any rectangular matrix
A^TA = atleast positive semi-definite or positive definite & symmetric.
* but positive definite $\Rightarrow$ symmetric
No need to explicitly mention

so A^TA is symmetric (+ve definite)

V are A^TA eigen vectors matrix (Orthogonal Matrix)

Σ^2 are eigen value matrix

Similarly

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T$$
$$AA^T = U\Sigma(V^T V)\Sigma^T U^T$$
$$AA^T = U\Sigma^2 U^T$$

where U is the orthogonal eigen vector matrix for AA^T

Σ^2 is the eigen value matrix for AA^T

AA^T is positive definite or atleast positive semidefinite.

Example $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

Step I:

Compute $A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

eigen values of $A^T A$ & eigen vector of $A^T A$

trial & error.

- $\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$ $\quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 32$

  $A\, x_1 = \lambda_1 x_1 \Rightarrow$

- Also $\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 18 \\ -18 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_2 = 18$

$$\Rightarrow V = \frac{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$$

Step II AA^T

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$$

$\lambda_1 = 32 \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 18$

$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A = U\Sigma V^T$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$ but $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

check $A = \begin{bmatrix} \sqrt{32}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ \sqrt{18}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & -3 \end{bmatrix}$? Next class

This change/Mistake will be discussed in next Review

Example 2

$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

row space = multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

column space = multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$$v_1 = \frac{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}{\sqrt{25}} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$u_1 = \frac{\begin{bmatrix} 4 \\ 8 \end{bmatrix}}{\sqrt{16+64}} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$v_2 =$ null space orthonormal basis

$$v_2 = \frac{\begin{bmatrix} 3 \\ -4 \end{bmatrix}}{\sqrt{25}} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}}$$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

since $|AA^T| = 0$ $\quad \lambda_1 = 0$

since trace $= 125$ $\quad \lambda_2 = 125$

check

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

A u Σ V^T

N(AT)

N(A^T) → multiplied by zeroes only so no effect but added to creat orthonormal matrix with non zero vector so that $V^TV = I$, $U^TU = I$

$$= \begin{bmatrix} 5 & 0 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

why we need N(A) & N(A^T) if they do not contribute?

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}_{\sqrt{5}} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}_{\sqrt{5}} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{\sqrt{5}}$$ not I

but $$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{\sqrt{5}\sqrt{5}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ I

or

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}_{\sqrt{5}\ \sqrt{5}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ Not I

$\{v_1 \ldots v_r\}$ orthonormal basis for row space
$\{u_1 \ldots u_r\}$ orthonormal basis for column space
$\{v_{r+1} \ldots v_n\}$ orthonormal basis for nullspace
$\{u_{r+1} \ldots u_m\}$ orthonormal basis for null space of A^T

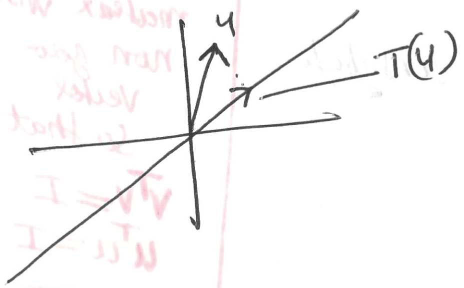
} for $A_{n \times m}$ for rectangular also
* SVD works.

and $Av_i = \sigma_i u_i$ — works even for Rectangular Matrices

Linear Transformation

Example 1: Projection

$T : R^2 \longrightarrow R^2$



Rules for Linear Transformation.

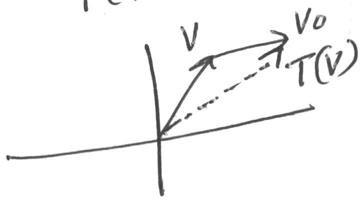
1. $T(v+w) = T(v) + T(w)$
2. $T(cv) = cT(v)$

1 & 2: $\boxed{T(c_1 v + c_2 w) = c_1 T(v) + c_2 T(w)}$

$\Rightarrow T(0) = 0$ vector

Non linear — Example 2: shift plane

$T(v) = v + v_0$



Non linear Transformation.

$T(2v) = 2v + v_0 \neq 2\,T(v)$

$T(0) = v_0$

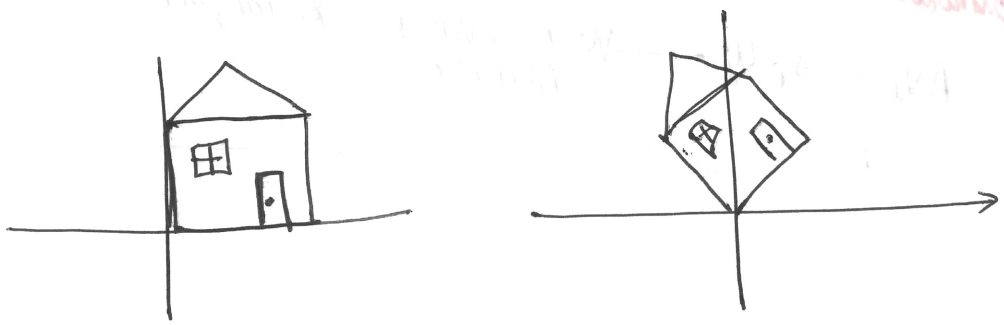
Ex 2

$T(v) = \|v\| \quad T : R^3 \longrightarrow R^1$

$T(-v) = T(v) \neq -T(v)$ ∴ Non Linear. Transformation.

Ex 3

$T(v) =$ rotate v by 45^0 $\quad T : R^2 \rightarrow R^2$ Linear.



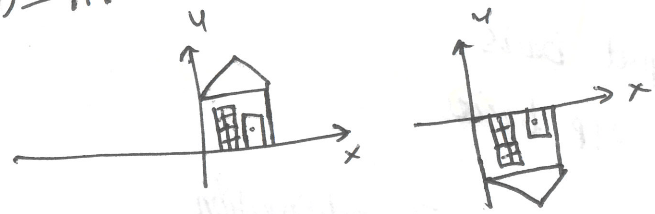
Example 4

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$T(V) = AV$

$\left.\begin{aligned} A(V+W) &= AV + AW \\ AcV &= cAV \\ A0 &= 0 \end{aligned}\right\}$ Linear

$T(V) = AV$



if $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Linear transformation.

* Now if T stands for Linear Transformation

$T: R^3 \to R^2$

$T(V) = AV = \begin{bmatrix} & & \end{bmatrix}_{2\times3} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} T_1(V) \\ T_2(V) \end{bmatrix}$

so $A = 2\times3$ Matrix

Information need to know $T(V)\ \forall\ V$.

$T(V_1), T(V_2)$, now we know $T(c_1V_1 + c_2V_2)$

$T(V_1), T(V_2) \ldots T(V_n)$ to know

so information needed $= T(V_1), T(V_2) \ldots T(V_n)$ to know
$T(V)\ \forall\ V \in$ basis $(V_1, V_2, \ldots V_n)$

* $T(V) = T(c_1V_1 + c_2V_2 + \ldots c_nV_n) = c_1T(V_1) + c_2T(V_2) + \ldots c_nT(V_n)$

Coordinates

- They come from a basis $\{V_1 \ldots V_n\}$
- Coordinates of $V = c_1V_1 + c_2V_2 + \ldots c_nV_n$
- They tell you how much of each basis vector is in V.
- Coordinates $= (c_1, c_2, c_3, \ldots c_n) \in R^n$

eg. standard cordinate system.

$$V = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Rectangular
x, y, z coordinates.

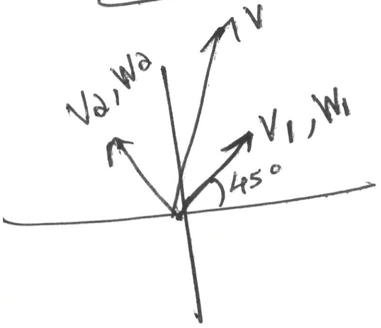
Construct the Matrix A that represent Linear transformation T

$T: R^n \rightarrow R^m \Rightarrow A_{m \times n}$

- Choose $\{v_1, \dots v_n\}$ the input basis
- Choose $\{w_1, w_2 \dots w_n\}$ the o/p basis

* WANT: Matrix A that does Linear Transformation

Example Projection



Matrix A representing projection on to vector v_1

Any i/p vector $v = c_1 v_1 + c_2 v_2$

$T(v) = c_1 v_1$ — only v_1 component is retained.

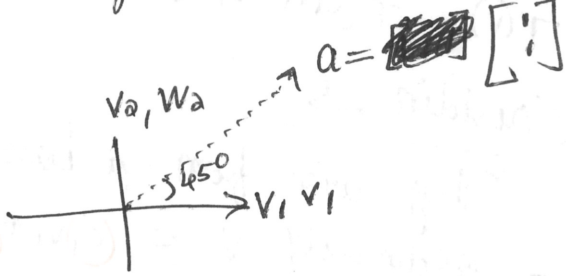
Cordinates of input $[c_1, c_2]$
Cordinates of o/p $[c_1, 0]$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\text{i/p cordinates}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{o/p cordinates}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$$

eigen vector basis is the good basis, it leads to diagonal matrix Λ.

* Same projection in standard basis

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

input $V = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}$

$\qquad\qquad\quad v_1 \qquad\quad v_2$

$w_1 = v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ basis

$w_2 \quad v_2 = \begin{bmatrix}0\\1\end{bmatrix}$

o/p $\quad W = \left(\frac{c_1+c_2}{2}\right)w_1 + \left(\frac{c_1+c_2}{2}\right)w_2$

$$A = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$$

Rule to find Matrix A. given Input Basis & o/p Basis.

1st Column of A : Apply $T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots a_{m1}w_m$

$$AV = \begin{bmatrix}a_{11} & & \\ a_{21} & & \\ \vdots & & \\ a_{m1} & & \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{matrix}c_1(A)\cdot R_1(V) + \longrightarrow c_1(A)\cdot R_1(V)\\ c_2(A)\cdot R_2(V) + \\ c_n(A) \quad R_n(V)\end{matrix}\Big\} 0$$

$\Rightarrow$ out coordinate $= (a_{11}, a_{21}, \dots a_{m1})$

$\Rightarrow$ output vector $= a_{11}w_1 + \dots a_{m1}w_m$

Similarly $C_2(A)$ $\quad T(v_2) = a_{12}w_1 + a_{22}w_2 + a_{32}w_3 + \dots a_{m2}w_m$

$\Rightarrow A\begin{pmatrix}\text{input}\\\text{coordinate}\\\text{vector}\end{pmatrix} = \begin{pmatrix}\text{output}\\\text{coordinates}\\\text{vector}\end{pmatrix}$

$T = \frac{d}{dx}$ $\quad$ Input : $c_1 + c_2x + c_3x^2$ $\quad$ basis $1, x, x^2$

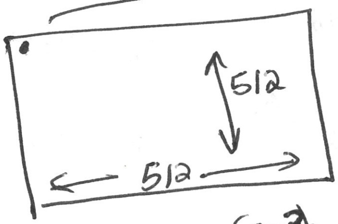
Ouput : $c_2 + 2c_3x$ $\quad$ basis $1, x,$ ~~x²~~

linear transformation

$$T(x) = \frac{d}{dx} = AV = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}c_2\\2c_3\end{bmatrix}$$

* Inverse matrix gives inverse transformation
* product of matrices gives product of Transformation.

Image Compression

$X =$ 

$X_I = 0 - 255$ integers or 8 bits

for grayscale image.

$X \in R^{(512)^2}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{bmatrix}$$

Standard compression JPEG : Joint photographic Experts group.

Standard basis

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

better basis.

Fourier basis 8x8.

$$\begin{bmatrix} 1 & 1 & \\ 1 & w & \\ 1 & w^2 & \\ \vdots & & \\ 1 & & \end{bmatrix}$$

Thresholding → throw away small coefficients

8x8

lossless Signal x — 64 bytes

↓ change basis

coefficient c ← 64 bytes

R^{64}

lossy ↓ compression eg:- Thresholding

coefficient $\hat{c}$ (Many zeroes)

$$\hat{x} = \underbrace{\sum \hat{c}_i v_i}_{\text{not all } 64}$$

Video sequence of images → highly correlated

Competetion for fourier

Wavelets ~~Basis~~ Basis in R8

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Standard Basis

Pixel values $P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix}$

want to write P as

$$P = \underbrace{\overset{\text{Wavelet } R^8}{\begin{bmatrix} w_1 & w_2 & \cdots & w_8 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}}_{\text{losseless step.}} \Rightarrow P = c_1 w_1 + c_2 w_2 + \cdots c_8 w_8$$

$\boxed{P = Wc}$ solve this equation to get c

$c = W^{-1}P$ solve this to get P to reproduce image

but $W^{-1} = W^T$ ~~bec~~ because W is orthogonal matrix

Good Basis

① Fast Multiplication: by W & W^{-1} is fast

* if it takes so much time then we cannot afford it inspite of the compression

FFT and Fast wavelet transform
- because basis vectors are orthogonal

} Remember transformation is just multiplication by the matrix & Inverse is multiplication by inverse matrix

② good Compression:
Few basis vectors are enough to reproduce image sufficiently at cost of quality.

Change of Basis

Columns of W = new basis vectors

$[x]$ is vector in old basis

$$\boxed{x = Wc}$$

$[x] \longrightarrow [c]$ new basis

Assume a Linear transformation T

w.r.t a basis $[v_1 \cdots v_8]$ it has matrix A

w.r.t a basis $[w_1 \cdots w_8]$ it has matrix B

Q What is connection of B to A.

old $X = Wc$ new

A: transformation T in basis $\{v_1 \dots v_8\}$

B: transformation T in basis $\{w_1 \dots w_8\}$

then A & B are similar

$\Rightarrow \quad B = M^{-1}AM$

proof

suppose X is the old vector in basis $\{v_1 \dots v_8\}$

suppose c is the same vector in new cordinate of basis $(w_1 \dots w_8)$

then T is defined as $T(X) = AX$ — basis $\{v_1 \dots v_8\}$

$T(c) = Bc$ — basis $\{w_1 \dots w_8\}$

We want to derive relation between A & B

We know $T(X)$ — in basis $\{v_1 \dots v_8\}$ $\Big\}$ * Here assuming the input & out are in same basis.

$T(c)$ — in basis $\{w_1 \dots w_8\}$

then $T(X) = WT(c) \longrightarrow$

$\Rightarrow AX = WBc$

$W^{-1}AX = Bc$

$W^{-1}AWc = Bc \quad [X = Bc]$

$\Rightarrow \underline{W^{-1}AW = B}$

* $X = Wc$

$X = x_1v_1 + x_2v_2 + \dots x_8v_8$

$c = c_1w_1 + c_2w_2 + \dots c_8w_8$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} w_{11} & & \\ w_{21} & & \\ \vdots & & \\ w_{81} & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ inputing w_1

Where C_1 of W represents ~~...~~ w_1 as the linear combination of $\{v_1 \dots v_8\}$

W is the cordinate transformation form c to X.

What is A? Using $\{v_1 \dots v_8\}$

Ans) I know T completely from ~~what~~ knowing what T does to each of $v_1, v_2, \dots v_8$

Because every X = some $c_1 \dots c_8$ such that

$X = c_1v_1 + \dots c_8v_8$

Then $T(X) = c_1T(v_1) + \dots c_8T(v_8)$

Write $T(v_1)$ as a combination of $\{v_1 \dots v_8\}$

ie $T(v_1)$ as $a_{11}(v_1) + a_{21}(v_2) + \dots a_{81}v_8$

$A = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{81} \\ \vdots & \vdots & & \\ a_{18} & a_{28} & & a_{88} \end{bmatrix}$

Suppose Eigen Vectors basis

ie $T(V_i) = d_i V_i$

What is A?

The choice of basis in which the Transformation T gives a vector o/p as multiple of input

Solution.

$$A = \begin{bmatrix} d_1 & 0 & 0 \cdots 0 \\ 0 & d_2 & 0 \cdots \\ \vdots & \vdots & d_3 \\ 0 & 0 & 0 \cdots d_n \end{bmatrix}$$

* With Signal processing it is prefered to look for eigen basis but that might be computationally costly. so they use standard basis.

First columns

$T(V_1) = d_1 (V_1)$ — transforming first basis vectors

$T(V_1) = d_1(V_1) + 0\, V_2 + \cdots 0 V_8$ — explexing as Linear combination of basis vectors.

$\Rightarrow a_{11} = d_1$

$a_{21} \cdots a_{81} \cdots = 0$

Similarly for all columns

* This selection of basis is ideal for image processing.
* But eigen vector calculation is too expensive.

Quiz Review

1) Eigen Values, & Eigen Vectors
2) $\frac{du}{dt} = Au$ and e^{At}
3) Symmetric Matrices $(Q \Lambda Q^T)$
4) Similar Matrix $B = M^{-1}AM \rightarrow$ same eigen values (B & A)
5) Positive definite Matrices
6) SVD: $A = U \Sigma V^T$

---

1) $\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u$

general solution.

$u(t) = c_1 e^{d_1 t} x_1 + c_2 e^{d_2 t} x_2 + c_3 e^{d_3 t} x_3$

eigen values & eigen vectors

$d_1 = 0$

$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\begin{vmatrix} -d & -1 & 0 \\ 1 & -d & -1 \\ 0 & 1 & -d \end{vmatrix} \Rightarrow -d\,[d^2+1] + 1(-d) \Rightarrow -d^3 - 2d = 0$

$-d(d^2 - 2) = 0$

$\rightarrow d = 0 \quad d^2 = 2 \quad d = \pm\sqrt{2}\, i$

$\Rightarrow \lambda_0 = 0 \quad \lambda_1 = +\sqrt{2}(i) \quad \lambda_2 = -\sqrt{2}(i)$

$[A - \lambda I]\, x = 0$

$$u(t) = c_1 e^{0t}[X_1] + c_2 e^{\sqrt{2}it} + c_3 e^{-\sqrt{2}it}$$

$$\begin{bmatrix} -\sqrt{2}i & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2}i \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 0$$

a) When does solution come back to initial value.

$e^{\sqrt{2}it}$ = same at $t=0$ & $t=t'$

$\Rightarrow e^0 = e^{\sqrt{2}it}$

$e^{2\pi i} = e^{\sqrt{2}it'} \Rightarrow \quad t' = \sqrt{2}\pi$

b)

$u(t) = e^{A(t)}\, u(0)$

If A can be diagonalised.

$e^{At} = S e^{\Lambda t} S^{-1}$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

Orthogonal Eign Vectors

Condition: $AA^T = A^TA$

only requirement

Family of matrix that satisfies the condition

- ✮ Symmetric matrix $A^T = A$
- ✮ Antisymmetric matrix $A^T = -A$
- ✮ Orthogonal (square) matrix $QQ^T = Q^TQ = I$

2) for A matrix $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = c \\ \lambda_3 = 2 \end{cases}$ $\quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

a) Which all c's allow this matrix A to be diagonalizable.

Solution all c's since eigen vectors are orthogonal

c) Which values of c's is matrix A symmetric

Sol) all real c

c) Positive definite.

~~5d)~~ all real $c > 0$ ✗ (it cannot be since one eigen vector is already 0.)

wrong

d) Markov Matrix

for $c = 1$ ✗

wrong

(No - markov Matrix is not a subset of Positive - but for markov matrix all eigen values must be less than ~~one~~ or equal to one)

****

$\boxed{\text{Projection Matrix subset of symmetric matrix}}$

e) $\frac{A}{2}$ (Projection matrix)?

Eigen values of projection matrixes are zero & 1.

So for A we need $\lambda = 0, 2$

So for $c = 0, 2$ we have a $\frac{A}{2}$ as projection matrix

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SVD

for every A (rectangular or square) $A = U \Sigma V^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$

$U^T U = I$ since U = orthogonal

$A^T A = V \Sigma^T U^T U \Sigma V^T$

$A^T A = V(\Sigma^T \Sigma) V^T$

orthogonal (diagonal square) orthogonal

$\Sigma^T \Sigma \neq \Sigma^2$ since Σ = diagonal ✗ since Σ may be rectangular diagonal

symmetric

→ V is eigen vector matrix for $A^T A$

$\sigma_i^2 = \lambda_i (A^T A)$ (where σ_i is the a_{ii}th entry in Σ)

also $AA^T = U \Sigma V^T V \Sigma^T U^T$

$AA^T = U(\Sigma \Sigma^T) U^T$

→ U is eigen vector matrix for AA^T

$\sigma_i^2 = \lambda_i (AA^T)$

In the ~~previous~~ last lecture when discussing SVD in an example a sign went wrong. This is the reason

but the signs of σ_i, eigen vector in V & eigen vectors in U are also bound by the relation

$AV = U\Sigma$ or $AV_1 = \sigma u_1$

* So when choosing sign should comply with $AV_1 = \sigma u_1$

Example

Q) Let $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ — if the 2 was (-5) they it is wrong the singular values are not negative

$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$

$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

but if $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ original matrix is also singular ~~sing~~ since Σ is singular

$A^{-1} = (U \Sigma V^T)^{-1} = V^{T^{-1}} \Sigma^{-1} U^{-1}$

Σ^{-1} does not exist

A^{-1} does not exist

$N(A) = \dim(1)$

$$\boxed{\begin{array}{l} \text{Vector in nullspace} = V_2 \\ \text{since } AV_2 = \sigma_2 u_2 \\ \quad \text{but } \sigma_2 = 0 \\ \Rightarrow AV_2 = 0 \\ \Rightarrow V_2 \in \text{Nullspace} \end{array}}$$

Given A = Symmetric & Orthogonal

$A^T = A$

$A^T A = I \Rightarrow A^2 = I \Rightarrow A^2 = I$

$A^T = A^{-1} \Rightarrow A = A^{-1} \Rightarrow A^2 = I$

$\not\Rightarrow$ $\boxed{A = I \text{ can have negative entries}}$

a) from this $|d_A| = \pm 1$

so A cannot be positive definite always since negative eigen value also posible.

b) No repeated eigen values: False

if 3x3 then repeated eigen values since $d = \pm 1$

c) Diagonlisable: True

All symmetrical matrix can be diagonlisable

d) Non Singular: True

Orthogonal $\Rightarrow$ Independent ~~eigen~~ column Vectors & row Vectors.

eigen ~~vect~~ values of orthogonal matrix

$|d_Q| = 1$

$Qx = dx$

$\|Qx\| = \|d\| \|x\|$

$\|x\| = \|d\| \|x\|$

$\Rightarrow |d| = 1$

orthogonal Matrixes dont change length.

e) ~~A~~ show $\frac{1}{2}(A+I)$ is a projection matrix

<u>for projection matrix</u>

$$\left.\begin{aligned} P:\ & P^2 = P \\ & P^n = P \quad \forall (n>1) \\ & P^T = P \end{aligned}\right\}$$

A symmetric
$\Rightarrow A+I$ symmetric
$\frac{A+I}{2}$ symmetric
$\Rightarrow \left(\frac{A+I}{2}\right)^T = \left(\frac{A+I}{2}\right)$

$$\frac{1}{2}^2(A+I)^2 = \frac{1}{4}(A^2+I^2+2AI) = \frac{1}{4}(I+I+2A) = \frac{1}{4}(2A+2I) = \frac{1}{2}(A+I)$$

$$\Rightarrow \left(\frac{1}{2}(A+I)\right)\left(\frac{1}{2}(A+I)\right) = \frac{1}{2}(A+I)$$

So by induction

$P^n = P \quad \forall\ n>0$

So $\frac{1}{2}(A+I)$ is a projection matrix

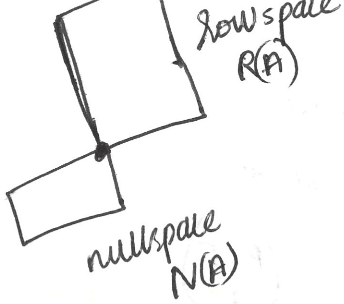
if $P^2 = P$
then $P^3 = P^2 \cdot P = P \cdot P = P^2 = P$

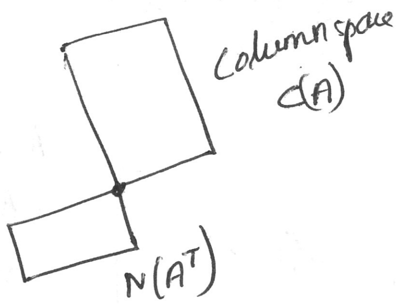
$$\Rightarrow \boxed{P^{n+1} = P \quad \text{if } P^n = P}$$

$$\boxed{\text{since } P^{n+1} = P^n \cdot P = P \cdot P = P}$$

What are eigen values of $\frac{1}{2}(A+I)$

$\Rightarrow$ <u>0,1</u> since λ of $A = \pm 1$
λ of $A+I = 0, 2$
λ of $\frac{A+I}{2} = 0, 1$





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2 sides inverse is what we call inverse.

$AA^{-1} = I = A^{-1}A$ $\quad$ $m=n$ No. of columns = No. of rows

$\quad$ rank $= n = m = r$ (Full rank)

Full column rank (left inverse)

$A_{m\times n}$ $\quad$ $r=n$ $\quad$ eg:- least square fitting problem

$m>n$ $\quad$ 0 or 1 solution to $Ax=b$ depending on whether b is in $C(A)$

$N(A) = \{0 \text{ vector}\}$

$A^TA =$ ~~n~~ $n\times n$ symmetric full rank matrix

$(A^TA)^{-1} A^T_{(n\times m)}$ = left side inverse of A where A = full column rank matrix.

proof: $\quad \underline{(A^TA)^{-1}A^TA = I}$ $\quad \Rightarrow \underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = I_{n\times n}$

Full row rank (right inverse)

$r=m$ $\quad$ $Ax=b$ is always solvable

$n>m$ $\quad$ but no unique solution

$N(A^T) = \{0 \text{ vector}\}$ $\quad$ $N(A) \neq \{0 \text{ vector only}\}$

because $m-n$ free variables.

$AA^T = m\times m$

$\boxed{A^T(AA^T)^{-1}}_{n\times m}$ right inverse.

$A_{m\times n}\ A^{-1}_{\substack{right\\ n\times m}} = I_{m\times m}$

assume A = full column rank $\quad$ Let $B = A^T$

then $(A^TA)^{-1}A^TA = I$

$\Rightarrow (BB^T)^{-1}BB^T = I$

$\Rightarrow BB^T = BB^T$

$I = BB^T(BB^T)^{-1}$

B = full ~~right~~ row rank $\quad$ $B^T(BB^T)^{-1}$ = right inverse of B

For full column rank A

$$AA^{-1}_{left} = A(A^TA)^{-1}A^T = P \text{ on the column space of } A$$

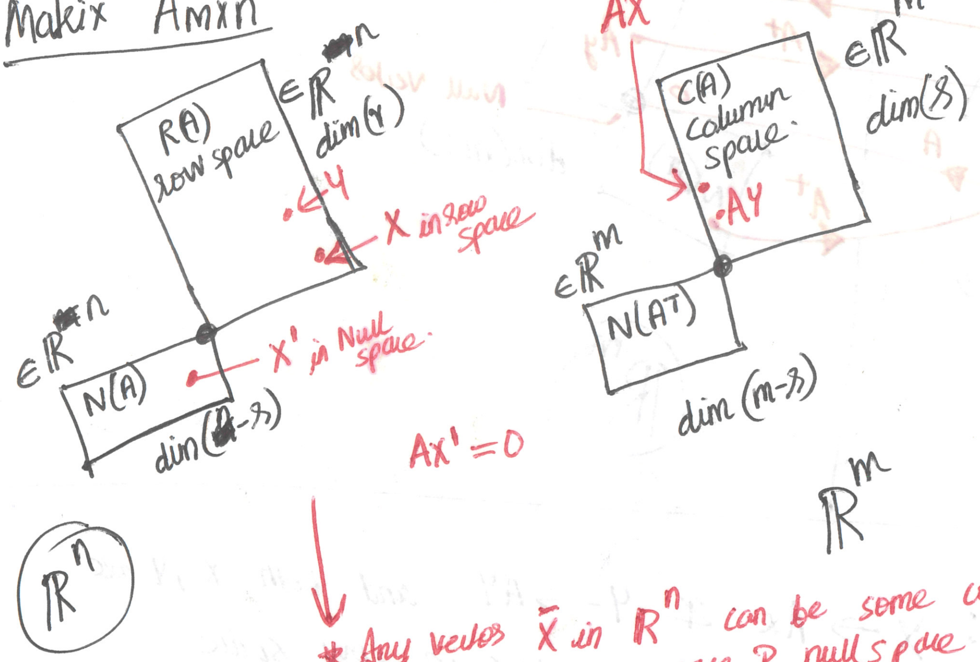
For full row rank A

$$A^{-1}_{right}A = A^T(AA^T)^{-1}A = P \text{ on the row space of } A$$

* "its those nullspaces that screw's up inverses, because if a matrix takes a vector to zero then there is no way to bring back the vector to life".

Psuedo inverses

Matrix $A_{m \times n}$



* Any vector $\bar{x}$ in R^n can be some combination of vectors in row space & nullspace $\bar{x} = x + x'$

$A\bar{x} = Ax + Ax'$

x and Ax has 1 to 1 correspondence

since $\left.\begin{array}{l} x \sim \text{Rowspace} \\ Ax \sim \text{Column space} \end{array}\right\}$ same dimension r.

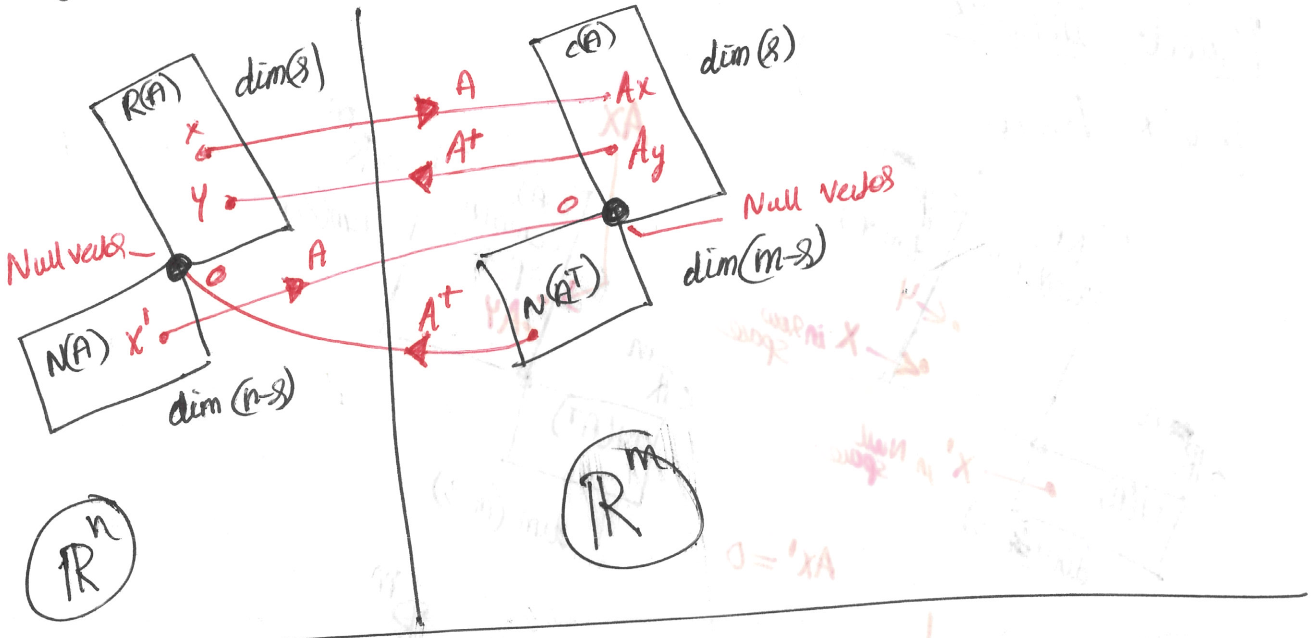
but $Ax' = 0$ Since $x' \in$ Nullspace of A

$A\bar{x} = Ax + 0$ → Then matrix A knock down the component of $\bar{x}$ with vectors in nullspace. Which cannot be recovered. Hence inverse does not exist.

If $x, \& y$ are in row space, the Ax and Ay are not equal.
ie every vector x in row space ($\mathbb{R}^n$, dim (r)) has one to one correspondence with ~~every~~ some vector in column space ($\mathbb{R}^m$, dim r) (since Ax in column space)

$\Rightarrow$ from row space column space A is invertible.
then the inverse is called pseudo inverse (A^+)

$$y = A^+(Ay)$$



Proof: Suppose : $x \to Ax$ & $y \to Ay$ and both x, y are non ~~z~~ zero and unequal vectors in row space.

if we assume $Ax = Ay$

$\Rightarrow A(x-y) = 0$ but $x - y \neq 0$ since x, y are not equal $\in R(A)$

$\Rightarrow (x-y) \in N(A)$ — since any non zero vector x such that $Ax = 0$ is in $N(A)$

but this can be true since both x & y are in $R(A)$
$x - y$ can't be in $N(A)$

which is contradicting since a vector can't be in $R(A)$ & $N(A)$ and not be zero vector.

$$= Ax \neq Ay \quad \forall \begin{cases} x, y \in R(A) \\ x \neq y \end{cases}$$

Least squares

* need A to be full column rank then they have left inverse & A^TA is invertible
* but the column may be dependent
  (A_{left} does not exist since A^TA becomes singular)

Find the pseudoInverse A^+

1) Start from SVD : $A = U\Sigma V^T$

$A = U\Sigma V^T$

$A^+ = V\Sigma^+ U^T$

$\left.\begin{aligned} U^{-1} &= U^T \\ V^{T-1} &= V \end{aligned}\right\}$ orthogonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}_{m\times n}$$

rank r
m rows
n colums.

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots \; 1/\sigma_r & \\ 0 & & \end{bmatrix}_{n\times m}$$

$$\Sigma\Sigma^+ = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots 0 \end{bmatrix}_{m\times m}$$

$$\Sigma^+\Sigma = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}_{n\times n}$$

Linear Algebra Review

$A_{m\times n}$ with rank r

1) given $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solution $\rightarrow$ column space does not span $\mathbb{R}^3 \Rightarrow r < m$

$\Rightarrow r < 3$

a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has 1 solution $\rightarrow$ $N(A) = \{0 \text{ vectors only}\}$

$\Rightarrow n = r$

$m = 3$
$n = ?$
$r < 3$
$n = r$

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [x] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Q 1b) True or false

1) $|A^TA| = |AA^T|$ —— False $|AA^T| = 0$ since singular

2) A^TA is invertible —— True because $(n=r)$ full column rank

3) AA^T is +ve definite —— False – only positive semidefinite

A^TA is invertible $N(A) = \{0 \text{ vectors}\}$

$\Rightarrow n = r$

$\Rightarrow$ Independent invertable.

$=$ full column rank(A)

$Ax = 0$

$A^TAx = 0$

$(A^TA)^{-1}(A^TA)x = 0$

$\Rightarrow x = 0$

$\Rightarrow N(A) = 0$ vector only

$AA^T = 3\times3$ and rank 2

non positive definite

AA^T – is only positive semidefinite.

1c) Prove that $A^Ty = C$ has atleast one solution for every C.

infinitely many solution for every C.

$A = n\times3$

$n = 1, 2$ full row rank $\Rightarrow$ always solution.

$\dim N(A^T) = m - r = 3 - r \geq 1$

because $\underline{r < 3}$.

---

2) $A = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix}$

a) Solve $Ax = V_1 - V_2 + V_3$

Solution. $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

b) Suppose that combination is zero then prove that solution is not unique.

because $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ is also solution

$N(A) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ span.

2c) Suppose v_1, v_2, v_3 are orthonormal what combination of v_1, v_2 is closest to v_3.

$\_v_1 + \_v_2 =$ closest to v_3 is $(0,0)$ because v_1, v_2, v_3 are perpendicular vectors

3.) Markov matrix $= A =$ $\begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$ $c_1 + c_2 = 2c_3$

$\lambda_1 = 0$ (because columns are dependent)
$\lambda_2 = 1$ (because it is Markov, they have one $\lambda = 1$)
$\lambda_3 = -0.2$ (to make trace right)

$u_k = A^k u_0$ if $u_0 = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ what's the solution as $k \to \infty$

$$\lim_{k\to\infty} u_k = c_1 \lambda_1^k \begin{bmatrix} x_1 \end{bmatrix} + c_2 \lambda_2^k \begin{bmatrix} x_2 \end{bmatrix} + c_3 \lambda_3^k \begin{bmatrix} x_3 \end{bmatrix}$$

$\lambda_1^k \to 0$, $\lambda_2^k \to 1$, $\lambda_3^k \to 0$

$$\lim_{k\to\infty} u_k = c_2 \begin{bmatrix} x_2 \end{bmatrix}$$

$[A - \lambda_2 I]\, x_2 = 0$

$$\Rightarrow \begin{bmatrix} -0.8 & 0.4 & 0.3 \\ 0.4 & -0.8 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = 0$$

$$x_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} k$$

$$\lim_{k\to\infty} u_k = c_2 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

Markov property is sum of population doesnt chang

$\Rightarrow c_2 = 1$

$$u_\infty = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

Tell me a matrix 2x2

a) Projection on to the line $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ Solve $P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}}{25}$

b) Matrix with $\lambda_1 = 0$, $\lambda_2 = 3$ $\quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = S\Lambda S^{-1}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 6 \\ -6 & 3 \end{bmatrix}$

c) $A \neq B^TB$ any B

B^TB is always symmetric

so A is any non symmetric

d) A has orthogonal eigen vectors but not symmetric matrix

~~A is Hermitian matrix.~~

A is orthogonal or skew symmetric

orthogonal eigen vectors

$\Rightarrow AA^T = A^TA$

$\Rightarrow$ symmetric $A^T = A$

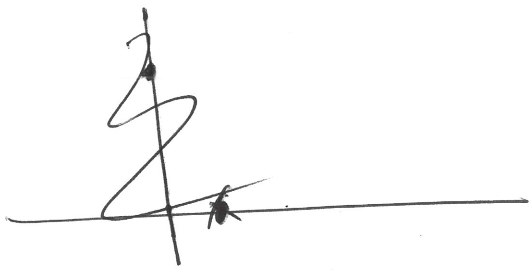
$\Rightarrow$ orthogonal $A^T = A^{-1}$

$\Rightarrow$ skew symmetric $A^T = -A$

5) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \leftarrow b$ $\qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 11/3 \\ -1 \end{bmatrix}$

best solution

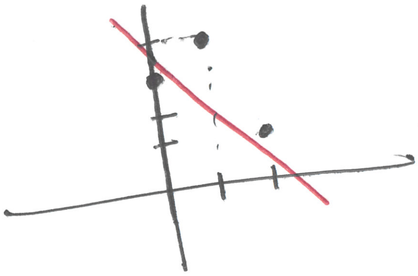
a) What is $Pb = \begin{bmatrix} -11/3 \\ -1 \end{bmatrix}$ $\qquad P = A(A^TA)^{-1}A^T$



b) Draw the problem

$C + Dx = y$ ⟶ straight line estimate.

~~at~~ point (0,3)
(1,4)
(2,1)



c) Find a real vector b for which the least square solution $\hat{P}b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

⟶ Problem ~~pro~~ statement modification

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix} \leftarrow b$$

$\Rightarrow b$ is orthogonal to ~~space~~ column space

$$b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \left. \begin{aligned} c_1^T b &= 0 \\ c_2^T b &= 0 \end{aligned} \right\} \Rightarrow b \perp \text{to columnspace}$$